7

ASSIGNMENT MODEL

7.1 INTRODUCTION

As already discussed earlier, linear programming relates to the problems concerning distributions of various resourses (such as *money, machines, time* etc.), satisfying some constraints which can be algebraically represented as linear *equations/inequalities* so as to *maximize profit* or *minimize cost.* This chapter deals with a very interesting method called the *'Assignment Technique'* which is applicable to a class of very practical problems generally called *'Assignment problems'*.

The name *'Assignment Problem'* originates from the classical problems where the objective is to assign a number of origins (jobs) to the equal number of destinations (persons) at a minimum cost (or maximum profit). To examine the nature of assignment problem, ***suppose there are n jobs to be performed and n persons are available for doing these jobs. Assume that each person can do each job at a time, though with varying degree of efficiency. Let c_{ij} be the cost (payment) if the ith person is assigned the jth job, the problem is to find an assignment (which job should be assigned to which person) so that the total cost for performing all jobs is minimum.*** Problems of this kind are known as *assignment problems.*

Table 7·1

Jobs

		1	2	...	j	...	n
	1	c_{11}	c_{12}	...	c_{1j}	...	c_{1n}
	2	c_{21}	c_{22}	...	c_{2j}	...	c_{2n}
Persons	:	:	:		:		:
	i	c_{i1}	c_{i2}	...	c_{ij}	...	c_{in}
	:	:	:		:		:
	n	c_{n1}	c_{n2}	...	c_{nj}	...	c_{nn}

Further, such types of problems may consist of assigning men to offices, classes to rooms, drivers to trucks, trucks to delivery routes, or problems to research teams, etc. The assignment problem can be stated in the form of $n \times n$ cost-matrix $[c_{ij}]$ of real number as given in Table 7·1 .

Q. 1. Define Assignment Problem.
2. What is an assignment problem ? **[IGNOU 2001, 99, 97, 96]**

7.2 MATHEMATICAL FORMULATION OF ASSIGNMENT PROBLEM

Mathematically, the assignment problem can be stated as :

$$\text{Minimize the total cost : } z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}, \; i = 1, 2, \ldots, n ; j = 1, 2, \ldots, n \qquad \ldots(7\cdot1)$$

subject to restrictions of the form :

$$x_{ij} = \begin{cases} 1 & \text{if } ith \text{ person is assigned } jth \text{ job} \\ 0 & \text{if not} \end{cases} \qquad \ldots(7\cdot2)$$

$$\sum_{j=1}^{n} x_{ij} = 1 \qquad (\text{one job is done by the } ith \text{ person}, i = 1, 2, \ldots, n) \qquad \ldots(7\cdot3)$$

and $\sum_{i=1}^{n} x_{ij} = 1$ (*only one person should be assigned the jth job,* $j = 1, 2, \ldots, n$) ...(7·4)

where x_{ij} *denotes that jth job is to be assigned to the ith person.*

This special structure of assignment problem allows a more convenient method of solution in comparison to *simplex method.*

7·2-1 Assignment Problem as a Special Case of Transportation Problem

The assignment problem (as defined in previous chapter) is seen to be the special case of transportation problem when each origin is associated with one and only one destination. In such a case, $m = n$ and the numerical evaluations of such association are called 'effectiveness' instead of 'transportation costs'. Mathematically, all a_i and b_j are unity, and each x_{ij} is limited to one of the two values 0 and 1. In such circumstances, exactly n of the x_{ij} can be non-zero (*i.e.* unity), one for each origin and one for each destination.

Q. 1. Give the mathematical formulation of an assignment problem.
[JNTU (B. Tech.) 2000; Meerut (Stat.) 98; Rewa (M.P.) 93 ; Meerut (IPM) 90]

2. Explain the conceptual justification that an assignment problem can be viewd as a linear programming problem.
[VTU (BE Mech.) 2002]

3. Give the mathematical formulation of transportation problem. How does it differ from an assignment problem ?
[VTU (BE Common) Feb. 2002; Meerut 2002; Madurai B.Sc. (Compu. Sc.) 92; Bharathidasan B.Sc. (Math) 90]

4. Explain the difference between a transportation problem and an assignment problem. Explain situations where an assignment problem can arise. **[JNTU (BE Comp. Sc.) 2004; Meerut (Maths) 99]**

5. Show that assignment problem is the special case of the transportation problem. **[IAS (Maths) 88]**

6. Give the mathematical formulation and difference between 'Transportation' and 'Assignment' problems.
[Agra 99; Kanpur 96; Meerut (IPM) 91; 90]

7.3 FUNDAMENTAL THEOREMS OF ASSIGNMENT PROBLEM

The solution to an assignment problem is fundamentally based on the following two theorems.

Theorem 7.1. ***Reduction Theorem :*** *In an assignment problem, if we add (or subtract) a constant to every element of a row (or column) of the cost matrix* $[c_{ij}]$, *then an assignment plan that minimizes the total cost for the new cost matrix also minimizes the total cost for the original cost matrix.*

[Meerut (Maths.) 96, 94, 93 P; Rewa (MP) 93]

Proof. Let $x_{ij} = X_{ij}$ minimizes the toal cost,

$$z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} \qquad \text{...(7·5)}$$

over all x_{ij} such that $x_{ij} \geq 0$ and $\sum_{i=1}^{n} x_{ij} = \sum_{j=1}^{n} x_{ij} = 1$. ...(7·6)

It is to be shown that the assignment $x_{ij} = X_{ij}$ also minimizes the new total cost

$$z' = \sum_{i=1}^{n} \sum_{j=1}^{n} (c_{ij} - u_i - v_j)\, x_{ij}$$

for all $i, j = 1, 2, \ldots, n$, where u_i and v_j are constants subtracted from ith row and jth column of the cost matrix $[c_{ij}]$. To prove this, it may be written as

$$z' = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} - \sum_{i=1}^{n} u_i \cdot \sum_{j=1}^{n} x_{ij} - \sum_{j=1}^{n} v_j \sum_{i=1}^{n} x_{ij}.$$

Using equations (7·5) and (7·6), we get

$$z' = z - \sum_{i=1}^{n} u_i - \sum_{j=1}^{n} v_j.$$

Since terms that are subtracted from z to give z' are independent of x_{ij}'s, it follows that z' is minimized whenever z is minimized, and conversely.

This completes the proof of the theorem.

Alternative statement of reduction theorem : *If* $x_{ij} = X_{ij}$ *minimizes* $z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$ *over all* $x_{ij} = 0$ *or* 1 *such that*

$$\sum_{i=1}^{n} x_{ij} = 1 \,,\ \sum_{j=1}^{n} x_{ij} = 1 \text{, then } x_{ij} = X_{ij}$$

also minimizes $$z' = \sum_{i=1}^{n} \sum_{j=1}^{n} c'_{ij} x_{ij} \text{, where } c'_{ij} = c_{ij} - u_i - v_j \text{ for } i, j = 1, 2, \dots, n \,.$$

and where u_i and v_j *are some real numbers.*

Corollary. *If* (x_{ij}), $i = 1, 2, \dots, n$; $j = 1, 2, \dots, n$ *is an optimal solution for an assignment problem with cost* (c_{ij}), *then it is also optimal for the problem with cost* (c_{ij}') *when*

$$c'_{ij} = c_{ij} \quad \text{for } i, j = 1, 2, \dots, n \,; j \neq k$$
$$c'_{ik} = c_{ik} - A \text{, where } A \text{ is a constant.}$$

Proof. We have

$$z' = \sum_i \sum_j c'_{ij} x_{ij} = \sum_i \Big(\sum_{j \neq k} c'_{ij} + c'_{ik} \Big) x_{ij} = \sum_i \Big(\sum_{j \neq k} c_{ij} + c_{ik} - A \Big) x_{ij} = \sum \sum c_{ij} x_{ij} - A \sum_i x_{ij}$$
$$= z - A \qquad (\text{since } \sum_i x_{ij} = 1)$$

Thus if (x_{ij}) minimizes z so will it z'.

Theorem 7.2. *In an assignment problem with cost* (c_{ij}), *if all* $c_{ij} \geq 0$ *then a feasible solution* (x_{ij}) *which satisfies* $\sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} = 0$, *is optimal for the problem.*

Proof. Since all $c_{ij} \geq 0$ and all $x_{ij} \geq 0$, the objective function $z = \sum \sum c_{ij} x_{ij}$ cannot be negative. The minimum possible value that z can attain is 0. Thus, any feasible solution (x_{ij}) that satisfies $\sum \sum c_{ij} x_{ij} = 0$ will be optimal.

Theorem 7.3. ***(König Theorem).*** *Let P be the set of 0 elements of a matrix C. Then the maximum number of 0's that can be selected in P such that no row or column of C contains more than one such 0 is equal to the minimum number of lines covering all the elements of P.*

Proof is beyond the scope of the book.

Corollary. *The maximal subset of P provides an optimal assignment when the minimum number of lines to cover all the elements of P is equal to the order of C.*

Proof. Left as an exercise for the reader.

Q. 1. Explain how an assignment problem can be treated as a linear programming problem. Show that the optimal solution to the assignment problem remains the same if a constant is added to or subtracted from any row or column of the cost matrix.

2. If $b_{ij} = c_{ij} - u_i - v_j$ $(i, j = 1, 1, 2, \dots, n)$ where u_i and v_j are constants, then show that an optimal solution of the assignment problem with cost matrix $B = (b_{ij})$ is also an optimal solution of the assignment problem with cost matrix $C = (c_{ij})$.

[Delhi B.Sc. (Maths.) 90]

7.4 HUNGARIAN METHOD FOR ASSIGNMENT PROBLEM

The solution technique of assignment problems can be easily explained by the following practical examples.

Example 1. *A department head has four subordinates, and four tasks have to be performed. Subordinates differ in efficiency and tasks differ in their intrinsic difficulty. Time each man would take to perform each task is given in the effectiveness matrix. How the tasks should be allocated to each person so as to minimize the total man-hours ?* **[JNTU (B. Tech) 2002, 2000; Tamil. (ERODE) 97; IAS (Main) 93; Kerala B.Sc. (Maths) 91; Meerut (Stat.) 90; Kalicut B. Tech 90]**

Table 7·2

Subordinates

		I	II	III	IV
Tasks	A	8	26	17	11
	B	13	28	4	26
	C	38	19	18	15
	D	19	26	24	10

Solution. To understand the problem initially, step by step solution procedure is necessary.

Step 1. Subtracting the smallest element in each row from every element of that row, we get the reduced matrix [*Table* 7·3]

Step 2. Next subtract the smallest element in each column from every element of that column to get the second reduced matrix [*Table* 7·4]

Table 7·3

0	18	9	3
9	24	0	22
23	4	3	0
9	16	14	0

Table 7·4

0	14	9	3
9	20	0	22
23	0	3	0
9	12	14	0

Step 3. Now, test whether it is possible to make an assignment using only *zeros*. If it is possible, the assignment must be optimal by *Theorem 7.2* of Section 7·3. Zero assignment is possible in *Table* 7·4 as follows:

(a) Starting with *row* 1 of the matrix (*Table* 7.4), examine the rows one by one until a row containing exactly *single zero* element is found. Then an experimental assignment (indicated by □) is marked to that cell. Now cross all other zeros in the *column* in which the assignment has been already made. This eliminates the possibility of marking further assignments in that column. The illustration of this procedure is shown in ***Table 7·5a.***

Table 7·5a

	I	II	III	IV
A	$\boxed{0}$	14	9	3
B	9	20	$\boxed{0}$	22
C	23	0	3	0
D	9	12	14	$\boxed{0}$

Table 7·5b

	I	II	III	IV
A	$\boxed{0}$	14	9	3
B	9	20	$\boxed{0}$	22
C	23	$\boxed{0}$	3	0
D	9	12	14	$\boxed{0}$

(b) When the set of rows has been completely examined, an identical procedure is applied sucessively to columns. Starting with *column* 1, examine all columns until a *column* containing exactly one zero is found. Then make an experimental assignment in that position and cross other zeros in the *row* in which the assignment has been made.

Continue these successive operations on rows and columns until all zeros have been either assigned or crossed-out. At this stage, re-examine rows. It is found that no additional assignments are possible. Thus, the complete '*zero assignment*' is given by A → I, B → III, C → II, D → IV as mentioned in *Table 7.5b*. According to *Theorem* 7.1, this assignment is also optimal for the original matrix (*Table* 7·2). Now compute the minimum total man-hours as follows:

Optimal assignment	:	A—I	B—III	C—II	D—IV	
Man-hour	:	8	4	19	10	(Total 41 hours.)

Now the question arises : *what would be further steps if the complete optimal assignment after applying Step 3 is not obtained ?* Such difficulty will arise whenever all zeros of any row or column are crossed-out. Following example will make the procedure clear.

Example 2. *A car hire company has one car at each of five depots a, b, c, d and e. A customer requires a car in each town, namely A, B, C, D, and E. Distance (in kms) between depots (origins) and towns (destinations) are given in the following distance matrix :*

Table 7·6

	a	*b*	*c*	*d*	*e*
A	160	130	175	190	200
B	135	120	130	160	175
C	140	110	155	170	185
D	50	50	80	80	110
E	55	35	70	80	105

How should cars be assigned to customers so as to minimize the distance travelled?

Solution. Applying *Step 1* and *Step 2* as explained in *Example 1* we get the *Table* 7.7.

Table 7·7

30	0	35	30	15
15	0	0	10	0
30	0	35	30	20
0	0	20	0	5
20	0	25	15	15

Step 3. Row 1 has a single zero in *column* 2. Make an assignment by putting a square '□' around it, and delete other zero (if any) in column 2 by marking '×'.

Table 7·8 a

15	$\boxed{0}$	35	30	15
15	$\cancel{0}$	$\boxed{0}$	10	$\cancel{0}$
15	$\cancel{0}$	35	30	20
$\boxed{0}$	$\cancel{0}$	20	$\cancel{0}$	5
5	$\cancel{0}$	25	15	15

Now, *column* 1 has a single zero in *row* 4. Make an assignment by putting '□' and cross the other zeros which is not yet crossed. Column 3 has a single zero in *row 2*, make an assignment and delete the other zeros which are uncrossed.

It is observed that there are no remaining zeros ; and *row 3, row 5, column 4,* and *column 5* each has no assignment. Therefore, desired solution cannot be obtained at this stage. we now, proceed to following important steps.

Step 4. Draw the minimum number of horizontal and vertical lines necessary to cover all zeros at least once. It should, however, be observed that (in all $n \times n$ matrices) less than n lines will cover zeros only when there is no solution among them. Conversely, if minimum number of lines is n, there is a solution. Following systematic procedure may help us to draw the minimum set of lines :

1. For simplicity, first make the *Table* 7·8*a* again and name it at *Table* 7·8 b.

Table 7·8 b

		L_1				
	30	$\boxed{0}$	35	30	15	✓ ④
L_2	15	$\cancel{0}$	$\boxed{0}$	10	$\cancel{0}$	
	30	$\cancel{0}$	35	30	20	✓ ①
L_3	$\boxed{0}$	$\cancel{0}$	20	$\cancel{0}$	5	
	20	$\cancel{0}$	25	15	15	✓ ②
		③✓				

2. Mark (√) *row 3* and *row 5* as they are having no assignments and *column 2* as having zeros in the marked rows 3 and 5.
3. Mark (√) *row 1* because this row contains assignment in the marked *column 2*. No further rows or columns will be required to mark during this procedure.
4. Now start drawing required lines as follows ;
   First draw line (L_1) through *marked* column 2. Then draw lines(L_2 and L_3) through *unmarked* rows (2 and 4) having largest number (2) of uncovered zeros (since no zero is left uncovered, the required lines will be (L_1, L_2 and L_3).

Step 5. In this step,

(i) first select the smallest element, say x, among all *uncovered* elements of the *Table* 7·8b [as a result of *step* 4] and

(ii) then subtract this value x from all values in the matrix not covered by lines and add x to all those values that lie at the intersection of any two of the lines L_1, L_2 and L_3. (Justification of this rule is given on the next page).

After applying these two rules, we find $x = 15$, and a new matrix is obtained as given in *Table 7·9*.

Table 7·9

15	0	20	15	0
15	15	0	10	0
15	0	20	15	5
0	15	20	0	5
5	0	10	0	0

Step 6. Now re-apply the test of *Step 3* to obtain the desired solution. Therefore, proceeding exactly in the same manner as in *step 3*, obtain the final *Table* 7·10.

Table 7·10

15	~~0~~	20	15	$\boxed{0}$
15	15	$\boxed{0}$	10	~~0~~
15	$\boxed{0}$	20	15	5
$\boxed{0}$	15	20	~~0~~	5
20	~~0~~	10	$\boxed{0}$	~~0~~

It is observed that there are no remaining zeros, and every row (column) has an assignment. Since no two assignments are in the same column (they cannot be, if the procedure has been correctly followed), the 'zero assignment' is the required solution.

From original matrix (*Table* 7·6), the minimum distance assignment is given by

Route	A – e	B – c	C – b	D – a	E – d	Total Distance Travelled
Distance (*Kms.*)	200	130	110	50	80	570 Kms.

Note. Table 7·10 may be obtained very quickly if we first apply *Step 2* and then *Step 1* in the original Table 14·6 ;

Justification of Rules Used Above in Step 5 :

Justification of rules we have used in *Step* 5 is based on the following two facts :

(i) The relative cost of assigning ith facility to jth job is not changed by the subtraction of a constant either from a column or from a row of the original effectiveness matrix.

(ii) An optimal assignment exists if the total reduced cost of the assignment is zero. This is the case when the minimum number of lines necessary to cover all zeros is equal to the order of the matrix. If however, it is less than n further reduction of the effectiveness matrix has to be undertaken.

The underlying logic can be explained with the help of ***Table 7·8(b)*** in which only 3 ($= n - 2$) lines can be drawn. Here an optimal assignment is not possible. So further reduction is necessary.

Further reduction is made by subtracting the smallest non-zero element 15 from all elements of the matrix ***Table 7.8(b)***. This gives the following matrix :

		L_1			
	15	−15	20	15	0
L_2	0	−15	15	−5	−15
	15	−15	20	15	0
L_3	15	−15	5	−15	−10
	5	−15	10	0	0

This matrix contains negative values. Since the objective is to obtain an assignment with reduced cost of zero, the negative numbers must be eliminated. This can be done by adding 15 to only those rows and columns which are covered by three *lines* (L_1, L_2, L_3) as shown above. In doing so the following change is noted.

		$L_1 \downarrow$			
	15	$(-15+15)$	20	15	0
$L_2 \rightarrow$	$(0+15)$	$[(-15+15)+15]$	$(-15+15)$	$(-15+15)$	$(-15+15)$
	15	$(-15+15)$	20	15	5
$L_3 \rightarrow$	$(-15+15)$	$[(-15+15)+15]$	$(5+15)$	$(-5+15)$	$(-10+15)$
	5	$(-15+15)$	10	0	0

This table is exactly the same as ***Table 7·9***. In fact, all this is the result of adding the least non-zero element at the intersections; and subtracting from all uncovered elements, and leaving the other elements unchanged.

Q. 1. Show that the procedure of subtracting the minimum elements not covered by any line, from all the uncovered elements and adding the same element to all the elements lying at the intersection of two lines results in a matrix with the same optimal assignments as the original matrix. **[Meerut M.Sc. (Math.) 92, 90; Jodhpur M.Sc. (Math) 92]**

2. State the assignment problem. Describe a method of drawing minimum number of lines in the context of assignment problem. Name the method.

3. Describe any method for solving an assignment problem. **[Delhi B.Sc. (Math.) 93]**

7·4-1 Assignment Algorithm (Hungarian Assignment Method)

Various steps of the computational procedure for obtaining an optimal assignment may be summarized as follows:

Step 1. Subtract the minimum of each row of the effectiveness matrix, from all the elements of the respective rows.

Step 2. Further, modify the resulting matrix by subtracting the minimum element of each column from all the elements of the respective columns. Thus obtain the *first modified matrix.*

Step 3. Then, draw the minimum number of horizontal and vertical lines to cover all the zeros in the resulting matrix. Let the minimum number of lines be N. Now there may be two possibilities:

(i) If $N=n$, the number of rows (columns) of given matrix, then an optimal assignment can be made. So make the zero assignment to get the required solution.

(ii) If $N<n$, then proceed to *step 4.*

Step 4. Determine the smallest element in the matrix, not covered by the N lines. Subtract this minimum element from all uncovered elements and add the same element at the intersection of horizontal and vertical lines. Thus, the *second modified matrix* is obtained.

Step 5. Again repeat ***Steps 3 and 4*** until minimum number of lines become equal to the number of rows (columns) of the given matrix *i.e.*, $N=n$.

Step 6. ***(To make zero-assignment)***. Examine the rows successively until a row-wise exactly single zero is found, mark this zero by '□' to make the assignment. Then, mark a cross (×) over all zeros if lying in the column of the marked '□' zero, showing that they cannot be considered for future assignment. Continue in this manner until all the rows have been examined. Repeat the same procedure for columns also.

Step 7. Repeat the ***Step 6*** successively until one of the following situations arise:

(i) if no unmarked zero is left, then the process ends; or

(ii) if there lie more than one of the unmarked zeros in any column or row, then mark '□' one of the unmarked zeros arbitrarily and mark a cross in the cells of remaining zeros in its row and column. Repeat the process until no unmarked zero is left in the matrix.

Step 8. Thus exactly one marked '□' zero in each row and each column of the matrix is obtained. The assignment corresponding to these marked '□' *zeros* will give the optimal assignment.

7·4-2 A Rule to Draw Minimum Number of Lines

A very convenient rule of drawing minimum number of lines to cover all the 0's of the reduced matrix is given in the follwing steps :

***Step* 1.** Tick (√) rows that do not have any marked (□) zero.

***Step* 2.** Tick (√) columns having marked (□) zeros or otherwise in ticked rows.

***Step* 3.** Tick (√) rows having marked 0's in ticked columns.

***Step* 4.** Repeat *steps 2 and 3* until the chain of ticking is complete.

***Step* 5.** Draw lines through all unticked rows and ticked columns.

This will give us the minimal system of lines.

Q. **1.** Give an algorithm to solve an 'Assignment Problem'. **[IGNOU 2001, 99, 97, 96; IAS (Maths) 88]**

2. Write a short note on 'Assignment Problem'.

3. Explain the Humgarian method to solve an assignment problem. **[Meerut (OR) 2003, 02; VTU (BE Mech.) 2002]**

7.5 MORE ILLUSTRATIVE EXAMPLES

Example 3. *Solve the assignment problem represented by the following matrix.* (*Table* 7·11).

[IAS (Maths.) 96; Meerut (B.Sc.) 90]

Table 7·11

	a	*b*	*c*	*d*	*e*	*f*
A	9	22	58	11	19	27
B	43	78	72	50	63	48
C	41	28	91	37	45	33
D	74	42	27	49	39	32
E	36	11	57	22	25	18
F	3	56	53	31	17	28

Solution. Step 1. Subtract the smallest element in each row from every element in that row to get the reduced matrix. (*Table 7·12*).

Table 7·12

0	13	49	2	10	18
0	35	29	7	20	5
13	0	63	9	17	5
47	15	0	22	12	5
25	0	46	11	14	7
0	53	50	28	14	25

Step 2. Subtract the smallest element in each column from every element in that column to get the second reduced matrix. (*Table* 7·13)..

Step 3. Make the 'zero-assignments' in usual manner. Since *row 2* and *column 5 of Table 7.13* have no assignments, go to next step.

Table 7·13

$\cancel{0}$	13	49	$\boxed{0}$	$\cancel{0}$	13
$\cancel{0}$	35	29	5	10	$\cancel{0}$
13	$\cancel{0}$	63	7	7	$\boxed{0}$
47	15	$\boxed{0}$	20	2	$\cancel{0}$
25	$\boxed{0}$	46	9	4	2
$\boxed{0}$	53	50	26	4	20

Step 4. Draw minimum number of lines to cover all zeros at least once. To do so, first mark (√) *row 2* as having no assignment and *columns* (*1* and *6*) as having zeros in row 2. Next, mark (√) the rows (3 and 6) as these two rows contain assignment in the marked columns (*1* and *6*).

Now draw lines L_1 and L_2 through each marked column (1 and 6), respectively. Then draw line L_3 through unmarked row 1 and line L_4 through unmarked column 2 (both having two uncovered zeros). Draw one more line L_5 either through unmarked row 4 or unmarked column 3. This way the minimum set of five lines (which is less than six) is obtained.

Table 7·14

	L_1	L_4				L_2	
L_3	0	13	49	$\boxed{0}$	~~0~~	13	
	0	35	29	5	10	0	√ ①
	13	0	63	7	7	$\boxed{0}$	√ ⑤
L_5	47	15	$\boxed{0}$	20	2	0	
	25	$\boxed{0}$	46	9	4	2	√ ⑦
	$\boxed{0}$	53	50	26	4	20	√ ④
	√ ②	√ ⑥				√ ③	

Step 5. The smallest element among all *uncovered* elements of *Table* 7·14 is 4. *Subtract* this value 4 from all values in *Table* 7·14 *not covered by lines*, and *add* 4 to all those values that lie at the intersection of the lines L_1, L_2, L_3, L_4 and L_5. Thus a new matrix (*Table 7·15a*) is obtained.

Step 6. Repeating the *Step 3*, make the 'zero assignments' as shown in *Table 7·15a*.

Table 7·15 a

	a	b	c	d	e	f
A	4	17	49	$\boxed{0}$	~~0~~	17
B	$\boxed{0}$	35	25	1	6	~~0~~
C	13	~~0~~	59	3	3	$\boxed{0}$
D	51	19	$\boxed{0}$	20	2	4
E	25	$\boxed{0}$	42	5	~~0~~	2
F	~~0~~	53	46	22	$\boxed{0}$	20

Here, it is also important that an assignment problem may have more than one solution. The other solution is shown in *Table 7·15b*.

Table 7·15 b

	a	b	c	d	e	f
A	4	17	49	$\boxed{0}$	~~0~~	17
B	~~0~~	35	25	1	6	$\boxed{0}$
C	13	$\boxed{0}$	59	3	3	~~0~~
D	51	19	$\boxed{0}$	20	2	4
E	25	~~0~~	42	5	$\boxed{0}$	2
F	$\boxed{0}$	53	46	22	~~0~~	20

These two solutions are :

(i) A → d, B → a, C → f, D → c, E → b and F → e ; (ii) A → d, B → f, C → b, D → c, E → e and F → a. with minimum cost z = **Rs. 142**.

Example 4. ***(Alternative Solutions).*** *Solve the minimal assignment problem whose effectiveness matrix is given by Table* 7·16. **[Meerut 2002]**

Table 7·16

	1	2	3	4
I	2	3	4	5
II	4	5	6	7
III	7	8	9	8
IV	3	5	8	4

Solution. *Step 1.* For each row in the matrix, subtract the smallest element in the row from each element in that row to get reduced matrix (*Table 7·17*)

Step 2. For each column in the reduced matrix, subtract the smallest element in the column from each element of that column to get the second reduced matrix *(Table 7·18)*

Table 7·17

0	1	2	3
0	1	2	3
0	1	2	1
0	2	5	1

Table 7·18

0	0	0	2
0	0	0	2
0	0	0	0
0	1	3	0

Step 3. Since single zeros neither exist in columns nor in rows, it is usually easy to make zero assignments. While examining rows successively, it is observed that *row 4* has two zeros in both the cells (4, 1) and (4, 4). Now, arbitrarily make an experimental assignment (indicated by □) to one of these two cells, say (4,1) and cross other zeros in *row 4* and *column 1*. ***Tables 7·19*a, *7·19*b** and ***7·19*c** show the necessary steps for reaching the optimal assignment : I → 2 , II → 3 , III → 4 , IV → 1 .

Table 7·19 a

	1	2	3	4
I	$\cancel{0}$	0	0	2
II	$\cancel{0}$	0	0	2
III	$\cancel{0}$	0	0	0
IV	$\boxed{0}$	1	3	$\cancel{0}$

Table 14·19 b

	1	2	3	4
I	$\cancel{0}$	0	0	2
II	$\cancel{0}$	0	0	2
III	$\cancel{0}$	$\cancel{0}$	$\cancel{0}$	$\boxed{0}$
IV	$\boxed{0}$	1	3	$\cancel{0}$

Table 7·19 c

	1	2	3	4
I	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	2
II	$\cancel{0}$	$\cancel{0}$	$\boxed{0}$	2
III	$\cancel{0}$	$\cancel{0}$	$\cancel{0}$	$\boxed{0}$
IV	$\boxed{0}$	1	3	$\cancel{0}$

Following other optimal assignments are also possible in this example.

$$\left\{\begin{array}{l} \text{I} \to 1\,,\ \text{II} \to 2\,,\ \text{III} \to 3\,,\ \text{IV} \to 4,\ \text{I} \to 3\,,\ \text{II} \to 2\,,\ \text{III} \to 1\,,\ \text{IV} \to 4 \\ \text{I} \to 3\,,\ \text{II} \to 2\,,\ \text{III} \to 4\,,\ \text{IV} \to 1,\ \ \text{I} \to 2\,,\ \text{II} \to 3\,,\ \text{III} \to 1\,,\ \text{IV} \to 4 \end{array}\right\} \text{(each having the cost 20)}$$

Example 5. ***(A Typical Problem)*** *An air-line operating seven days a week has time-table shown below. Crews must have a minimum layover (rest) time of 5 hrs between flights. Obtain the pair of flights that minimizes layover time away from home. For any given pair, the crew will be based at the city that results in the smaller layover. For each pair, mention the town where the crews should be based.*

Delhi-Jaipur			Jaipur-Delhi		
Flight No.	Depart	Arrive	Flight No.	Depart	Arrive
1	7.00 A.M	8.00 *A.M.*	101	8.00 A.M.	9.15 A.M.
2	8.00 A.M.	9.00 *A.M.*	102	8.30 A.M.	9.45 A.M.
3	1.30 P.M.	2.30 *P.M.*	103	12.00 Noon	1.15 P.M.
4	6.30 P.M.	7.30 *P.M.*	104	5.30 P.M.	6.45 P.M.

[Meerut (M.Sc. Math) 2002, 99, 98, 96BP]

Solution. *Step 1.* Construct the table for layover times between flights when crew is based *at Delhi.* For simplicity, consider 15 minutes = 1 unit.

Table 7·20 . Layover times when crew based at Delhi

		Flights 101	102	103	104
	1	96	98	112	38
Flights	2	92	94	108	34
	3	70	72	86	108
	4	50	52	66	88

Since the crew have a minimum layover of 5 hrs between flights, the layover time between flights 1 and 101 will be 24 hrs (96 units) from 8.00 *A.M.* to 8.00 *A.M.* next day.

Likewise, calculate as follows :

Flight No.	Layover Times	No. of Units (1 hr. = 4 units)
1 → 102	8 *A.M.* – 8.30 *A.M.* = 24 hrs 30 min	98
1 → 103	8 *A.M* – 12 *Noon* = 28 *hrs*	112
1 → 104	8 *A.M.* – 5.30 *P.M.* = 9 hrs 30 min	38
2 → 101	9.00 *A.M.* – 8 *A.M.* = 23 hrs	92

and so on.

Similarly, layover times for other pair of flights can also be calculated as shown in *Table 7·20*.

Step 2. Similarly, construct the table for layover times between flights when crew is based at *Jaipur*.

Table 7·21. Layover times when crew based at Jaipur

Flights → ↓	101	102	103	104
1	87	85	71	49
2	91	89	75	53
3	113	111	97	75
4	37	35	21	95

Since the plane arrives *Delhi* at 9·15 *A.M.* by flight number 101 and again depart to *Jaipur* at 7·00 *A.M.* by flight number 1, the layover time is obviously 21 hrs 45 min (*i.e.* 87 units). Similarly, layover times between other pairs of flight can also be computed as shown in *Table 7·21*.

Step 3. Construct the table for smaller layover times between flights with the help of ***Tables*** ***7·20*** and ***7·21***. Layover times marked '*' denote that the crew is based at *Jaipur*. Thus ***Table 7·22*** is obtained.

Table 7·22. Smaller layover times

	101	102	103	104
1	87*	85*	71*	38
2	91*	89*	75*	34
3	70	72	86	75*
4	37*	35*	21*	88

Step 4. Finally applying the assignment technique in the usual manner, we get the *Table 7·23*.

Table. 7.23

	101	102	103	104
1	4*	$\cancel{0}$	$\boxed{0^*}$	$\cancel{0}$
2	12*	8*	8*	$\boxed{0}$
3	$\boxed{0}$	$\cancel{0}$	28	50*
4	4*	$\boxed{0^*}$	$\cancel{0}^*$	100

From *Table* 7·23 the optimal assignments are : (3 – 101), (4 – 102)*, (1 – 103)*, (2 – 104) which gives the minimum layover time of 52 hr. 30 min.

Example 6. *A certain equipment needs five repair jobs which have to be assigned to five machines. The estimated time (in hours) that each mechanic requires to complete the repair job is given in the following table* :

Job / Machine	J_1	J_2	J_3	J_4	J_5
M_1	7	5	9	8	11
M_2	9	12	7	11	10
M_3	8	5	4	6	9
M_4	7	3	6	9	5
M_5	4	6	7	5	11

Assuming that each mechanic can be assigned to only one job, determine the minimum time assignment.

(Rajasthan (M. Com.) 97)

Solution. ***Step 1.*** Subtracting the smallest element of each row from all the elements of that row and then subtracting the smallest element of each column from all the elements of that column, we get the reduced matrix as shown in ***Table 7.24*** and ***Table 7.25*** respectively.

Table 7.24

Machine \ Job	J_1	J_2	J_3	J_4	J_5
M_1	2	0	4	3	6
M_2	2	5	0	4	3
M_3	4	1	0	2	5
M_4	4	0	3	6	2
M_5	0	2	3	1	7

Table 7.25

Machine \ Job	J_1	J_2	J_3	J_4	J_5
M_1	2	0	4	2	4
M_2	2	5	0	3	1
M_3	4	1	0	1	3
M_4	4	0	3	5	0
M_5	0	2	3	0	5

Step 2. Now we attempt to make a complete set of assignments using only a single zero element in each row or column. Since row M_1 contains only single zero, the assignment is made in the cell (M_1, J_2) and the zero appearing in the corresponding column J_2 is crossed out. Similarly, the assignment is made in the cell (M_2, J_3) and the zero appearing in the corresponding column J_4 is crossed out. Now row M_4 has only single zero, therefore the assignment is made in cell (M_4, J_5). Since there are two zeros in row M_5, we cannot make assignment in this row M_5. Looking columnwise, we find that column J_1 has only single zero, therefore we make an assignment in cell (M_5, J_1) and cross out the zero appearing in the corresponding row M_5. The assignments so made are shown in ***Table 7.26.***

Table 7.26

Machine \ Job	J_1	J_2	J_3	J_4	J_5
M_1	2	$\boxed{0}$	4	2	4
M_2	2	5	$\boxed{0}$	3	1
M_3	4	1	$\cancel{0}$	1	3
M_4	4	$\cancel{0}$	3	5	$\boxed{0}$
M_5	$\boxed{0}$	2	3	$\cancel{0}$	5

Thus, it is possible to make only four of the five necessary assignments using the zero element position. We, therefore, create one more zero element by drawing the minimum number of horizontal and vertical lines. Usually the minimum number of lines to cover all the zeros can be obtained by inspection. However, we shall use the method given earlier in explaining the various steps. The various steps for drawing the minimum number of lines are :

(a) Mark the row M_3 which has no assignment.

(b) Mark column J_3 which has zero in the marked row M_3.

(c) Mark row M_2 which has assignment in marked column J_3,

(d) Repeat steps (a) and (b) until no more rows or columns can be marked.

(e) Draw the lines through unmarked rows and marked columns.

The minimum number of lines drawn are shown in ***Table 7.27.*** It must be checked that the number of lines drawn are equal to the number of assignments made. But we require five assignments. To create one more zero, we examine the elements not covered by these lines and select the smallest element, *viz.*, 1 from among these uncovered lines.

Table 7.27

Machine \ Job	J_1	J_2	J_3	J_4	J_5
M_1	2	$\boxed{0}$	4	2	4
M_2	2	5	$\boxed{0}$	3	1
M_3	4	1	$\cancel{0}$	1	3
M_4	4	$\cancel{0}$	3	5	$\boxed{0}$
M_5	$\boxed{0}$	2	3	$\cancel{0}$	5

Subtract this smallest element 1 from all the uncovered elements and add it to the element where the two lines intersect. The reduced matrix so obtained is shown in adjoining *Table 7.28.* Proceeding in the usual way, the set of assignments made are shown in ***Table 7.28.***

Table 7.28

Machine \ Job	J_1	J_2	J_3	J_4	J_5
M_1	2	$\boxed{0}$	5	2	4
M_2	1	4	$\boxed{0}$	2	$\cancel{0}$
M_3	3	$\cancel{0}$	$\cancel{0}$	$\boxed{0}$	2
M_4	4	$\cancel{0}$	4	5	$\boxed{0}$
M_5	$\boxed{0}$	2	4	$\cancel{0}$	5

The optimum solution is :

Assign Job	To Machine	Cost (Rs.)
M_1	J_2	5
M_2	J_3	7
M_3	J_4	6
M_4	J_5	5
M_5	J_1	4
	Minimum total cost	= Rs. 27

Example 7. *ABC company is engaged in manufacturing 5 brands of packed snaks. It is having five manufacturing setups, each capable of manufacturing any of its brands, one at a time. The cost to make a brand on these setups vary according to the following table :*

	S_1	S_2	S_3	S_4	S_5
B_1	4	6	7	5	11
B_2	7	3	6	9	5
B_3	8	5	4	6	9
B_4	9	12	7	11	10
B_5	7	5	9	8	11

Assuming, five setups are $S_1, S_2, S_3, S_4,$ and S_5 *and five brands are* B_1, B_2, B_3, B_4 and B_5. *Find the optimum assignment of products on these setups resulting in the minimum cost.* **(C.A. Nov., 98)**

Solution. ***Step 1*** : Select the minimum element in each row and substract this element from every element in the row to get ***table 7.29.***

Table 7.29

	S_1	S_2	S_3	S_4	S_5
B_1	0	2	3	1	7
B_2	4	0	3	6	2
B_3	4	1	0	2	5
B_4	2	5	0	4	3
B_5	2	0	4	3	6

Step 2 : Select the minimum element in each column and subtract this element from every element in the column to get the ***table 7.30.***

Table 7.30

	S_1	S_2	S_3	S_4	S_5
B_1	0	2	3	0	5
B_2	4	0	3	5	0
B_3	4	1	0	1	3
B_4	2	5	0	3	1
B_5	2	0	4	2	4

Step 3 : We can attempt to make a complete set of assignments using only a single zero element in each row or each column. Since row B_5 contains a single zero, thus the assignment is ($B_5 \rightarrow S_2$). The other zero appearing in column S_2 is crossed out. Similarly the other assignments are made. Only four assignments can be made at this stage.

Table 7.31

	S_1	S_2	S_3	S_4	S_5
B_1	$\boxed{0}$	2	3	$\cancel{0}$	5
B_2	4	$\cancel{0}$	3	5	$\boxed{0}$
B_3	4	1	$\cancel{0}$	1	3
B_4	2	5	$\boxed{0}$	3	1
B_5	2	$\boxed{0}$	4	2	4

Step 4 : Since only four assignments could be made hence one more zero element need to be created by drawing the minimum number of horizontal and vertical lines. Mark the row B_3 which has no assignment. Mark column S_3 which has a zero in the marked row B_3. Mark row B_4 which has assignment in marked column S_3. Draw the lines through unmarked rows and marked columns to get ***table 7.32.***

Table 7.32

	S_1	S_2	S_3	S_4	S_5	
B_1	$\boxed{0}$	2	3	0	5	
B_2	4	0	3	5	$\boxed{0}$	
B_3	4	1	~~0~~	1	3	✓
B_4	2	5	$\boxed{0}$	3	1	✓
B_5	2	$\boxed{0}$	4	2	4	
			✓			

Step 5 : To get number of lines drawn (*i.e.*, 4) equal to number of assignments to be made (*i.e.*, 5), we still need one more line. To create one more zero, the elements not covered by these lines are examined and smallest among them is selected which is '1'. Subtract this smallest element from every uncovered elements and add it to the element where the horizontal and vertical lines intersect to get the ***Table 7.33.***

Table 7.33

	S_1	S_2	S_3	S_4	S_5
B_1	$\boxed{0}$	2	4	0	5
B_2	4	0	4	5	$\boxed{0}$
B_3	3	0	0	$\boxed{0}$	2
B_4	1	4	$\boxed{0}$	2	0
B_5	2	$\boxed{0}$	5	2	4

The optimum assignment is :

Assign Brand	To set up	Cost (Rs.)
B_1	S_1	4
B_2	S_5	5
B_3	S_4	6
B_4	S_3	7
B_5	S_2	5

Minimum total cost = Rs. 27

Alternative solution after Step 2 :

Step 3 : Draw the minimum number of lines to cover all zeros as shown in adjoining ***Table 7.34.***

Table 7.34

	S_1	S_2	S_3	S_4	S_5
B_1	0	2	3	0	5
B_2	4	0	3	5	0
B_3	4	1	0	1	3
B_4	2	5	0	3	1
B_5	2	0	4	2	4

Step 4 : Since the minimum number of lines to cover all zeros is 4 which is one less than the order of the matrix (= 5), the above table will not give the optimum solution. Subtract the minimum uncovered element (= 1) from all uncovered elements and add it to the elements lying at the intersection of two lines to get ***Table 7.35.***

Table 7.35

	S_1	S_2	S_3	S_4	S_5
B_1	0	3	4	0	5
B_2	4	1	4	5	0
B_3	3	1	0	0	2
B_4	1	5	0	2	0
B_5	1	0	4	1	3

Step 5 : The minimum number of lines to cover all zeros is 5 which is equal to the order of the matrix. The above table will give the optimum assignment as shown in ***Table 7.35.***

Example 8. *Five men are available to do five different jobs. From past records, the time (in hours) that each man takes to do each job is known and given in the following table :*

		Job I	II	III	IV	V
	A	2	9	2	7	1
	B	6	8	7	6	1
Man	C	4	6	5	3	1
	D	4	2	7	3	1
	E	5	3	9	5	1

Find the assignment of men to jobs that will minimize the total time taken.

(A.I.M.A. (P.G. Dip. In Management) Dec. 95)

Solution. *Step 1* : Subtracting the smallest element of each row from every element of the corresponding row, we get the adjoining reduced matrix (*Table* 7.36).

Table 7.36

Man \ Job	I	II	III	IV	V
A	1	8	1	6	0
B	5	7	6	5	0
C	3	5	4	2	0
D	3	1	6	2	0
E	4	2	8	4	0

Step 2 : Subtract the smallest element of each column from every element of the corresponding column to get the adjoining reduced matrix (*Table* 7.37).

Table 7.37

Man \ Job	I	II	III	IV	V
A	0	7	0	4	0
B	4	6	5	3	0
C	2	4	3	0	0
D	2	0	5	0	0
E	3	1	7	2	0

Step 3 : Row 2 has a single zero in column 5. Make an assignment by putting square (□) around it, and delete the other zeros in column 5 by marking '×'.

Now, column 4 has a single zero in row 3. We make an assignment by putting (□) and cross the other zeros which is not yet crossed. Column 2 has a single zero in row 4, we make an assignment.

Table 7.38

Man \ Job	I	II	III	IV	V
A	$\boxed{0}$	7	$\cancel{0}$	4	1
B	3	5	4	2	$\boxed{0}$
C	2	4	3	$\boxed{0}$	1
D	2	$\boxed{0}$	5	$\cancel{0}$	1
E	2	$\cancel{0}$	6	1	$\cancel{0}$

It may be noted that there are no remaining zeros, and row *E* and column *III* has no assignment. Thus, the optimum solution is not reached at this stage and we proceed to the following important steps.

Step 4 : Draw the minimum number of horizontal and vertical lines necessary to cover all zeros at least once. The following systematic procedure may help to draw the minimum set of lines :

(i) For simplicity, first make the table 7.39, again.

Table 7.39

Man \ Job	I	II	III	IV	V	
A	$\boxed{0}$	7	$\cancel{0}$	4	$\cancel{0}$	
B	4	6	5	3	$\boxed{0}$	✓
C	2	4	3	$\boxed{0}$	$\cancel{0}$	✓
D	2	$\boxed{0}$	5	$\cancel{0}$	$\cancel{0}$	
E	3	1	7	2	$\cancel{0}$	
					✓	

(ii) Secondly, mark (√) row 5 in which there is no assignment, *i.e.*, the last row.

(iii) Then mark (√) column 5, which has a zero in the marked row.

(iv) Next mark (√) row 2, which has assignment in the marked column.

(v) Draw the minimum number of lines covering the unmarked rows and the marked columns.

Step 5 : Examine the elements that do not have a line through them. Select the smallest of these and subtract it from all the elements that do not have a line through them. Add this element to every element lying at the intersection of two lines. Leave the remaining elements of the matrix unchanged.

Table 7.40

Man \ Job	I	II	III	IV	V
A	$\boxed{0}$	7	$\cancel{0}$	4	1
B	3	5	4	2	$\boxed{0}$
C	2	4	3	$\boxed{0}$	1
D	2	$\boxed{0}$	5	$\cancel{0}$	1
E	2	$\cancel{0}$	6	1	$\cancel{0}$

Step 6 : Repeat the steps to obtain optimum solution.

Thus in ***Table 7.41***, there are no remaining zeros, and every row and column has assignment, optimum solution is reached.

Hence the minimum time taken = 2 + 1 + 4 + 3 + 3 = 13 hours.

Table 7.41

Man \ Job	I	II	III	IV	V
A	$\cancel{0}$	9	$\boxed{0}$	6	3
B	1	5	2	2	$\boxed{0}$
C	$\boxed{0}$	4	1	$\cancel{0}$	1
D	$\cancel{0}$	$\cancel{0}$	3	$\boxed{0}$	1
E	$\cancel{0}$	$\boxed{0}$	4	1	$\cancel{0}$

Example 9. *An automobile dealer wishes to put four repairmen to four different jobs. The repairmen have somewhat different kinds of skills and they exhibit different levels of efficiency from one job to another. The dealer has estimated the number of manhours that would be required for each job-man combination. This is given in the matrix form in adjacent table :*

Man \ Job	A	B	C	D
1	5	3	2	8
2	7	9	2	6
3	6	4	5	7
4	5	7	7	8

Find the optimum assignment that will result in minimum manhours needed. **(Madras (M. Com.) 98)**

Solution. ***Steps 1***. Subtracting the smallest element of each row from all the elements of that row and then in the second matrix subtracting the smallest element of each column from all the elements of that column, the initial feasible solution determined by zeros is obtained.

Table 7.42

Man \ Job	A	B	C	D
1	3	1	0	3
2	5	7	0	1
3	2	0	1	0
4	0	2	2	0

Steps 2. We now examine each row successively for a single zero. Enrectangling these zeros and crossing (×) all the remaining zeros in the respective columns and then repeating same procedure for each column, the adjacent ***Table 7.43*** is obtained.

Table 7.43

Man \ Job	A	B	C	D
1	3	1	$\boxed{0}$	3
2	5	7	$\cancel{0}$	1
3	2	$\boxed{0}$	1	$\cancel{0}$
4	$\boxed{0}$	2	2	$\cancel{0}$

But still optimum assignment is not reached, since no zero has been marked in the *second* row and *fourth* column.

Steps 3. We now draw the minimum number of horizontal and vertical lines necessary to cover all zeros at least once. This has been obtained in ***Table 7.44***. Select the smallest element not covered by the lines (*i.e.*, 1) and subtract it from all the elements not covered by the lines and add the same to the elements at the intersection of the lines. We thus obtain ***Table 7.45*** providing the second feasible solution to the problem.

Table 7.44

Man \ Job	A	B	C	D	
1	3	1	$\boxed{0}$	3	
2	5	7	$\cancel{0}$	1	
3	2	$\boxed{0}$	1	$\cancel{0}$	L_2
4	$\boxed{0}$	2	2	$\cancel{0}$	L_3
			L_1		

Table 7.45

Man \ Job	A	B	C	D
1	2	0	0	2
2	4	6	0	0
3	2	0	2	0
4	0	2	3	0

Step 4. Repeating step 2, we make the 'zero-assignments' as shown in the ***Table 7.46 a***. It may be noted that an assignment problem can have more than one optimum solution. The other solution is shown in ***Table 7.46 b***.

Table 7.46 a

Man \ Job	A	B	C	D
1	2	$\boxed{0}$	$\cancel{0}$	2
2	4	6	$\boxed{0}$	$\cancel{0}$
3	2	$\cancel{0}$	2	$\boxed{0}$
4	$\boxed{0}$	2	3	$\cancel{0}$

Optimum solution I

Man	Job	Man hours
1	B	3
2	C	2
3	D	7
4	A	5

Table 7.46 b

Man \ Job	A	B	C	D
1	2	$\cancel{0}$	$\boxed{0}$	2
2	4	6	$\cancel{0}$	$\boxed{0}$
3	2	$\boxed{0}$	2	$\cancel{0}$
4	$\boxed{0}$	2	3	$\cancel{0}$

Optimum solution II

Man	Job	Man hours
1	C	2
2	D	6
3	B	4
4	A	5

Example 10. *A private firm employs typists on hourly piece rate basis for their daily work. Five typists are working in that firm and their charges and speeds are different. On the basis of some earlier understanding, only one job is given to one typist and the typist is paid for full hours even when he or she works for a fraction of an hour. Find the least cost allocation for the following data :*

Typist	Rate per hour (Rs.)	Number of pages typed prr hr.
A	5	12
B	6	14
C	3	8
D	4	10
E	4	11

Job	No. of pages
P	199
Q	175
R	145
S	298
T	178

[C.A., (Nov.) 96; Delhi (M.B.A.) Nov. 96]

Solution.

Step 1. The following matrix gives the cost incurred if the ith typist $(i = A, B, C, D, E)$ executes the j th job $(j = P, Q, R, S, T)$.

Table 7.47

Typist \ Job	P	Q	R	S	T
A	85	75	65	125	75
B	90	78	66	132	78
C	75	66	57	114	69
D	80	72	60	120	72
E	76	64	56	112	68

Steps 2. Subtracting the minimum element of each row from all its elements in turn, the adjoining matrix reduces to :

Table 7.48

Typist \ Job	P	Q	R	S	T
A	20	10	0	60	10
B	24	12	0	66	12
C	18	9	0	57	12
D	20	12	0	60	12
E	20	8	0	56	12

Now subtract the minimum element of each column from all its elements, in turn, the above matrix reduces to

Table 7.49

Typist \ Job	P	Q	R	S	T
A	2	2	0	4	0
B	6	4	0	10	2
C	0	1	0	1	2
D	2	4	0	4	2
E	2	0	0	0	2

Step 3. Since there are only 4 lines (< 5) to cover all zeros, optimum assignment cannot be made. The minimum uncovered element is 2. We, therefore subtract the minimum uncovered element '2' from all uncovered elements, add this value to all junction values and leave the other elements undisturbed, as shown in the adjoining matrix;

Table 7.50

Typist \ Job	P	Q	R	S	T
A	2	2	2	4	0
B	4	2	0	8	0
C	0	1	2	1	2
D	0	2	0	2	0
E	2	0	2	0	2

Step 4. Since the minimum number of lines required to cover all the zeros is only 4 (< 5), optimum assignment cannot be made at this stage also. The minimum uncovered element is 1. Repeating the usual procedure again, we get the adjoining matrix :

Table 7.51

Typist \ Job	P	Q	R	S	T
A	2	1	2	3	0
B	4	1	0	7	0
C	0	0	2	0	2
D	0	1	0	1	0
E	3	0	3	0	3

Step 5. Since the minimum number of lines to cover all zeros is equal to 5, this matrix will give optimum solution. The optimum assignment is made in the matrix given below :

Table 7.52

Typist \ Job	P	Q	R	S	T
A	2	1	2	3	$\boxed{0}$
B	4	1	$\boxed{0}$	7	$\cancel{0}$
C	$\cancel{0}$	$\boxed{0}$	2	$\cancel{0}$	2
D	$\boxed{0}$	1	$\cancel{0}$	1	$\cancel{0}$
E	3	$\cancel{0}$	3	$\boxed{0}$	3

Typist		Job		Cost (Rs.)
A	→	*T*	—	75
B	→	*R*	—	66
C	→	*Q*	—	66
D	→	*P*	—	80
E	→	*S*	—	112
			Total	Rs. 399

Remark. It may be noted that the above solution is not unique as a alternate optimum solution also exists.

EXAMINATION PROBLEMS

1. Solve the following assignment problems

(a) Man

Work	1	2	3	4
I	12	30	21	15
II	18	33	9	31
III	44	25	24	21
IV	23	30	28	14

(b) Jobs

Operators	I	II	III	IV
A	10	12	9	11
B	5	10	7	8
C	12	14	13	11
D	8	15	11	9

[**Ans.** (a) I → 1 , II → 3 , III → 2 , IV → 4 , min cost = 60 ; (b) A → III , B → I , C → II , D → IV , min cost = 37 .]

2. Solve the following cost-minimizing problems :

(a) Jobs

	I	II	III	IV	V
A	45	30	65	40	55
B	50	30	25	60	30
C	25	20	15	20	40
D	35	25	30	30	20
E	80	60	60	70	50

(b)

Machines	I	II	III	IV	V
A	2	9	2	7	1
B	6	8	7	6	1
C	4	6	5	3	1
D	4	2	7	3	1
E	5	3	9	5	1

[Delhi B.Sc. (Maths.) 90]

(c)

		Jobs				
		I	II	III	IV	V
	A	11	10	18	5	9
	B	14	13	12	19	6
Machines	C	5	3	4	2	4
	D	15	18	17	9	12
	E	10	11	19	6	14

[Delhi B.Sc. (Maths.) 93, 89]

[Ans. (a) (A → III, B → V, C → I, D → IV, E → II) or (A → III, B → V, C → IV, D → I, E → II) or (A → III, B → V, C → IV, D → II, E → I), min cost = 13.
(b) (A → II, B → III; C → I, D → IV, E → V) or (A → II, B → III, C → IV, D → I, E → V), min cost = 160
(c) A → II, B → V, C → III, D → IV, E → I, and min cost = 39]

3. A team of 5 horses and 5 riders has entered a jumping show contest. The number of penalty points to be expected when each rider rides any horse is shown below:

Horses	Riders				
	R_1	R_2	R_3	R_4	R_5
H_1	5	3	4	7	1
H_2	2	3	7	6	5
H_3	4	1	5	2	4
H_4	6	8	1	2	3
H_5	4	2	5	7	1

How should the horses be alloted to the riders so as to minimize the expected loss of the team.
[Ans. $H_1 \to R_5$, $H_2 \to R_1$, $H_3 \to R_4$, $H_4 \to R_3$, $H_5 \to R_2$; min. loss = 8.]

4. A company has six jobs to be done on six machines; any job can be done on any machine. The time in hours taken by the machines for the different jobs are as given below. Assign the machines to jobs so as to minimize the total machine hours.

Machines	Jobs					
	1	2	3	4	5	6
1	2	6	7	3	8	7
2	6	1	3	9	7	3
3	3	6	5	7	3	5
4	2	2	7	8	4	8
5	4	9	6	8	7	6
6	7	5	5	7	7	5

[Delhi B.Sc. (Math.) 91]

[Ans. 1 → 4, 2 → 2, 3 → 5, 4 → 1, 5 → 3, 6 → 6 or 1 → 4, 2 → 2, 3 → 5, 4 → 1, 5 → 6, 6 → 3 or 1 → 4, 2 → 2, 3 → 5, 4 → 2, 5 → 1, 6 → 6 ; min time = 20 hrs.]

5. A small aeroplane company, operating seven days a week serves three cities A, B, and C according to the schedule shown in the following table. The layover cost per stop is roughly proprotional to the square of the layover time. How should planes be assigned to the flights so as to minimize the total layover cost?

Flight No. and Index	From	Departure	To	Arrival
A_1B	A	09 *AM*	B	*Noon*
A_2B	A	10 *AM*	B	01 PM
A_3B	A	03 *PM*	B	06 PM
A_4C	A	08 *PM*	C	Mid. Night
A_5C	A	10 *PM*	C	02 AM
B_1A	B	04 *AM*	A	07 AM
B_2A	B	11 *AM*	A	02 PM
B_3A	B	03 *PM*	A	06 PM
C_1A	C	07 *AM*	A	11 AM
C_2A	C	03 *PM*	A	07 PM

[Agra 98]

[Ans.

Flight No :	1	2	3	4	5
Departure Route :	A_1B	A_2B	A_3B	C_1A	C_2A
Arrival Route :	B_3A	B_1A	B_2A	A_4C	A_5C]

6. A trip from Madurai to Trivandrum takes 6 hours by bus. A typical time-table of bus services in both directions is given below :

Madurai-Trivendrum			Trivandrum-Madurai		
Route No.	Depart	Arrive	Route No	Depart	Arrive
a	06·00	12·00	1	05·30	11·30
b	07·30	13·30	2	09·00	15·00
c	11·30	17·30	3	15·00	21·00
d	19·00	01·00	4	18·30	00·30
e	00·30	06·30	5	00·00	06·00

The cost of providing this service by the transport company depends upon the time spent by the bus crew (driver and conductor) away from their places in addition to service times. There are five crews. There is a constraint that every crew should be provided with 4 hours of rest before return trip again and should not wait for more them 24 hours for the return trip. The company has residential facilities for the crew at Madurai as well as at Trivendrum. Obtain the pairing of routes so as to minimize the cost. **[VTU (BE Mech.) 2002; Madurai 93]**

[Ans.

Crew :	1	2	3	4	5
Residence :	Madurai	Trivendrum	Trivendrum	Madurai	Trivendrum
Service No. :	d1	2e	3a	b4	5c
Waiting Time (hrs) :	4·5	9·5	9·0	5·0	5·5

Minimum total waiting time = 33·30 hours.]

7.6 UNBALANCED ASSIGNMENT PROBLEM

If the cost matrix of an assignment problem is not a square matrix (number of sources is not equal to the number of destinations), the assignment problem is called as *Unbalanced Assignment Problem.* In such cases, *fictitious* rows and/or columns with zero costs are added in the matrix so as to form a square matrix. Then the usual assignment algorithm can be applied to this resulting balanced problem.

Q. What is an unbalanced assignment problem ? **[IGNOU 2001, 99, 97, 96]**

Example 11. *A company is faced with the problem of assigning six different machines to five different jobs. The costs are estimated as follows (in hundreds of rupees) :*

Table 7·53

Jobs

		1	2	3	4	5
	1	2.5	5.0	1.0	6	1.0
	2	2.0	5.0	1.5	7	3.0
Machines	3	3.0	6.5	2.0	8	3.0
	4	3.5	7.0	2.0	9	4.5
	5	4.0	7.0	3.0	9	6.0
	6	6.0	9.0	5.0	10	6.0

Solve the problem assuming that the objective is to minimize the total cost.

Solution. Introducing one more column for a fictitious job (say, job 6) in the cost matrix in order to get the following balanced assignment problem. The cost corresponding to sixth column are always taken as zero.

Table 7·54

Jobs

		1	2	3	4	5	6 (Dummy)
	1	2·5	5·0	1·0	6	1·0	0
	2	2·0	5·0	1·5	7	3·0	0
Machines	3	3·0	6·5	2·0	8	3·0	0
	4	3·5	7·0	2·0	9	4·5	0
	5	4·0	7·0	3·0	9	6·0	0
	6	6·0	9·0	5·0	10	6·0	0

Since the problem can be solved as in usual practice, there is no need to give the detailed solution here, because it has already been explained earlier.

Example 12. *A Methods Engineer wants to assign four new methods to three work centres. The assignment of the new methods will increase production and they are given below. If only one method can be assigned to a work centre, determine the optimum assignment :*

Increase in production (unit)

		Work centres		
		A	B	C
Methods	1	10	7	8
	2	8	9	7
	3	7	12	6
	4	10	10	8

Solution. The given problem is of maximization type, since the elements of the given matrix relate to increase in production of units due to introduction of new methods. First of all, convert it into minimization problem by subtracting each element of the given matrix from maximum element 12. Since the problem is unbalanced one, introduce a dummy work centre.

Table 7.55

Method \ Work centre	A	B	C	Dummy
1	2	5	4	0
2	4	3	5	0
3	5	0	6	0
4	2	2	4	0

Subtracting the smallest element of each column from all elements of that column, we get the adjoining *Table 7.56.*

Table 7.56

Method \ Work centre	A	B	C	Dummy
1	0	5	0	0
2	2	3	1	0
3	3	0	2	0
4	0	2	0	0

Since the minimum number of horizontal and vertical lines to cover up all zeros is 4, the reduced matrix will give the optimum solution.

Table 7.57

Method \ Work centre	A	B	C	Dummy
1	$\boxed{0}$	5	0	0
2	2	3	1	$\boxed{0}$
3	3	$\boxed{0}$	2	0
4	0	2	$\boxed{0}$	0

The allocations as obtained from the above process are $1 \to A$, $2 \to$ Dummy, $3 \to B$, $4 \to C$. The total production under the above assignment is :

$$10 \text{ units} + 12 \text{ units} + 8 \text{ unit} = 30 \text{ units}.$$

Example 13. *A city corporation has decided to carry out road repairs on main four arteries of the city. The government has agreed to make a special grant of Rs. 50 lakhs towards the cost with a condition that the repairs must be done at the lowest cost and quickest time. If conditions warrant, then a supplementary token grant will also be considered favourably. The corporation has floated tenders and 5 contractors have sent in their bids. In order to expedite work, one road will be awarded to only one contractor.*

Cost of repairs (Rs. lakhs)

		R_1	R_2	R_3	R_4
	C_1	9	14	19	15
	C_2	7	17	20	19
Contractors/Road	C_3	9	18	21	18
	C_4	10	12	18	19
	C_5	10	15	21	16

(i) Find the best way of assigning the repair work to the contractors and the costs.
(ii) If it is necessary to seek supplementary grants, then what should be amount sought ?
(iii) Which of the five contractors will be unsuccessful in his bid ?

Solution. Since this is an assignment problem with 5 contractors and 4 roads, a dummy road 'R_5' with zero cost of repairing for each contractor is introduced to make the problem balanced.

Step 1 (a) Row subtraction

Contractors \ Road	R_1	R_2	R_3	R_4	R_5
C_1	9	14	19	15	0
C_2	7	17	20	19	0
C_3	9	18	21	18	0
C_4	10	12	18	19	0
C_5	10	15	21	16	0

Step 1 (b) Column subtraction

Contractors \ Road	R_1	R_2	R_3	R_4	R_5
C_1	2	2	1	0	0
C_2	0	5	2	4	0
C_3	2	6	3	3	0
C_4	3	0	0	4	0
C_5	3	3	3	1	0

Step 2. Draw minimum straight lines to cover all zeros.

2	2	1	0	0
0	5	2	4	0
2	6	3	3	0
3	0	0	4	0
3	3	3	1	0

Step 3. Smallest uncovered number is then subtracted from uncovered numbers added to numbers at intersection of two lines.

2	2	1	0	1
0	5	2	4	1
1	5	2	2	0
3	0	0	4	1
2	2	2	0	0

Step 4. Return to *step* 2. cover all zeros, since the number of lines is 4, the optimality criteria is not satisfied.

1	1	0	0	1
0	5	2	5	2
0	4	1	2	0
3	0	0	5	2
1	1	1	0	0

Step 5. Return to *step* 3. All rows and columns have single allocation and hence optimality criteria is satisfied.

1	1	$\boxed{0}$	$\cancel{0}$	1
$\boxed{0}$	5	2	5	2
$\cancel{0}$	4	1	2	$\boxed{0}$
3	$\boxed{0}$	$\cancel{0}$	5	2
1	1	1	$\boxed{0}$	$\cancel{0}$

Thus allotments are as follows

(i) $R_1 \xrightarrow{7} C_2$, $R_2 \xrightarrow{12} C_4$, $R_3 \xrightarrow{19} C_1$, $R_4 \xrightarrow{16} C_5$, $R_5 \xrightarrow{0} C_3$, Total cost = Rs. 54 lacs

(ii) Since cost exceeds 50 lacs, the excess amount of Rs. 4 (54 – 50) lacs is to be sought as supplimentary grant.

(iii) Contractor C_3 who has been assigned the dummy row (Road R_5) loses out in the bid.

EXAMINATION PROBLEMES

Q. 1. A company has 4 machines on which to do 3 jobs. Each job can be assigned to one and only one machine. The cost of each job on each machine is given in the following table :

		Machine W	Machine X	Machine Y	Machine Z
Job	A	18	24	28	32
	B	8	13	17	19
	C	10	15	19	22

What are the job assignments which will minimize the cost ? **[Gauhati (M.C.A.) 92]**
[Ans. A → W , B → X , C → Y ; or A → W , B → Y , C → X ; min cost = 50]

2. In a machine shop, a supervisor wishes to assign five jobs among six machines. Any one of the jobs can be processed completely by any one of the machines as given below :

		Machine A	B	C	D	E	F
Job	1	13	13	16	23	19	9
	2	11	19	26	16	17	18
	3	12	11	4	9	6	10
	4	7	15	9	14	14	13
	5	9	13	12	8	14	11

The assignment of jobs to machines be on a one-to-one basis. Assign the jobs to machines so that the total cost is minimum. Find the minimum total cost. **[I.A.S. (Maths.) 98]**
[Ans. 1 → F, 2 → A, 3 → E, 4 → C. 5 → D, min cost = 43 .]

3. A department head has six jobs and five subordinates. The subordinates differ in their efficiency and the tasks differ in their intrinsic difficulty. The department head estimates the time each man would take to perform each task as given in the effectiveness matrix below :

		Task A	B	C	D	E	F
Man	1	20	15	26	40	32	12
	2	15	32	46	26	28	20
	3	11	15	2	12	6	14
	4	8	24	12	22	22	20
	5	12	20	18	10	22	15

Only one task can be assigned to one man. Determine how should the jobs be allocated so as to minimize the total man hours. Find the minimum total man hours.
[Ans. 1 → F, 2 → A, 3 → E, 4 → C, 5 → D ; or 1 → B , 2 → F , 3 → C, 4 → A, 5 → D ; min time = 55 hrs .]

4. A truck company on a particular day has 5 trucks for sending material to 6 terminals. The cost of sending material from some destination to different trucks will be different as given by the cost matrix below. Find the assignment of 4 trucks to 4 terminals out of six at the minimum cost.

		Trucks A	B	C	D	E
Terminals	1	3	6	2	6	5
	2	7	1	4	4	7
	3	3	8	5	8	3
	4	6	4	3	7	4
	5	5	2	4	3	2
	6	5	7	6	2	5

[Ans. 1 → C, 2 → B, 3 → A, 6 → D ; min cost = 8]

5. Solve the following unbalanced assignment problem of minimizing total time for doing all the jobs :

		Job 1	2	3	4	5
Operator	1	6	2	5	2	6
	2	2	5	8	7	7
	3	7	8	6	9	8
	4	6	2	3	4	5
	5	9	3	8	9	7
	6	4	7	4	6	8

[Ans. 1 → 4, 2 → 1, 3 → dummy 6, 4–5, 5–2, 6 → 3, min. time = 16 units]

6. To stimulate interest and provide an atmosphere for intellectual discussion a finance faculty in a management school decides to hold special seminars on four contemporary topics–leasing, portfolio management, private mutual funds, swaps and options. Such seminars should be hold once per week in the afternoons. However, scheduling these seminars (one for each topic, and not more than one seminar per afternoon) has to be done carefully so that the number of students unable to attend is kept to a minimum. A careful study indicates that the number of students who cannot attend a particular seminar on a specific day is as follows :

	Leasing	Portfolio management	Private Mutual funds	Swaps and options
Monday	50	40	60	20
Tuesday	40	30	40	30
Wednesday	60	20	30	20
Thursday	30	30	20	30
Friday	10	20	10	30

Find an optimal schedule of the seminars. Also find-out the total number of students who will be missing at least one seminar. **[C.A. (Nov.) 92]**

[Ans. Optimal Schedule : **No. of Students Missing**

Monday	: Swaps and options	20
Tuesday	: No. seminar	0
Wednesday	: Partfolio management	20
Thursday	: Private mutual funds	20
Friday	: Leasing	10
	Total	70

7.7 VARIATIONS IN THE ASSIGNMENT PROBLEM

In this section, we shall discuss two variations of the assignment problem.

7·7-1 The Maximal Assignment Problem

Sometimes, the assignment problem deals with the maximization of an objective function rather than to minimize it. For example, it may be required to assign persons to jobs in such a way that the expected profit is maximum. Such problem may be solved easily by first converting it to a minimization problem and then applying the usual procedure of assignment algorithm. This conversion can be very easily done by subtracting from the highest element, all the elements of the given profit matrix; or equivalently, by placing minus-sign before each element of the profit-matrix in order to make it cost-matrix.

Q. 1. When in an assignment problem, the objective function is of maximization instead of minimization, what modifications are needed in the assignment algorithm to achieve this maximal assignment ?
2. What other variations of an assignment problem are possible ?
3. How can you maximize an objective function in the assignment problem.

Following examples will make the procedure clear.

Example 7. ***(Maximization Problem).*** *A company has 5 jobs to be done. The following matrix shows the return in rupees on assigning ith (i = 1, 2, 3, 4, 5) machine to the jth job (j = A, B, C, D, E). Assign the five jobs to the five machines so as to maximize the total expected profit.*

Jobs

		A	B	C	D	E
	1	5	11	10	12	4
	2	2	4	6	3	5
Machines	3	3	12	5	14	6
	4	6	14	4	11	7
	5	7	9	8	12	5

[Kerala B.Sc. (Math.) 90]

Solution. ***Step 1. Converting from Maximization to Minimization :***

Since the highest element is 14, so subtracting all the elements from 14, the following reduced cost (opportunity loss of maximum profit) matrix is obtained.

9	3	4	2	10
12	10	8	11	9
11	2	9	0	8
8	0	10	3	7
7	5	6	2	2

Step 2. Now following the usual procedure of solving an assignment problem, an optimal assignment is obtained in the following table :

1	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	5
$\cancel{0}$	13	$\cancel{0}$	5	$\boxed{0}$
5	1	7	$\boxed{0}$	5
3	$\boxed{0}$	9	4	5
$\boxed{0}$	3	3	1	5

This table gives the optimum assignment as : $1 \to C$, $2 \to E$, $3 \to D$, $4 \to B$, $5 \to A$;
with maximum profit of Rs. 50.

Example 15. ***(Maximization Problem).*** *A company has four territories open, and four salesmen available for assignment. The territories are not equally rich in their sales potential ; it is estimated that a typical salesman operating in each territory would bring in the following annual sales :*

Territory	: I	II	III	IV
Annual sales (Rs.)	: 60,000	50,000	40,000	30,000

Four salesmen are also considered to differ in their ability : it is estimated that, working under the same conditions, their yearly sales would be proportionately as follows :

Salesman	:	*A*	*B*	*C*	*D*
Proportion	:	7	5	5	4

If the criterion is maximum expected total sales, then intuitive answer is to assign the best salesman to the richest territory, the next best salesman to the second richest, and so on. Verify this answer by the assignment technique. **[Meerut 2004; VTU (BE Comp. Sc.) Aug. 2002]**

Solution. ***Step 1. To construct the effectiveness matrix :***

In order to avoid the fractional values of annnual sales of each salesman in each territory, it will be rather convenient to consider the sales for 21 years (the sum of proportions : $7 + 5 + 5 + 4 = 21$), taking Rs. 10,000 as one unit. Divide the individual sales in each territory by 21, if the annual sales by salesman are required.

Thus, the sales matrix for maximization is obtained as follows :

Table 7·58

Sales in 10 thousand of rupees	6	5	4	3
Sales proportion ↓	I	II	III	IV
7 A	42	35	28	21
5 B	30	25	20	15
5 C	30	25	20	15
4 D	24	20	16	12

Step 2. ***(To convert 'the maximum sales matrix' to 'minimum sales matrix'.)***

The problem of 'maximization' can be converted to 'minimization' one, by simply multiplying each element of given matrix (*Table 7·58*) by −1. Thus resulting matrix becomes :

Table 7·59

	I	II	III	IV
A	−42	−35	−28	−21
B	−30	−25	−20	−15
C	−30	−25	−20	−15
D	−24	−20	−16	−12

Step 3. Subtracting the smallest element in each row from every element in that row, we get the reduced matrix (*Table 14·60*).

Table 7·60

0	7	14	21
0	5	10	15
0	5	10	15
0	4	8	12

Step 4. Subtract the smallest element in each column from every element in that column to get the second reduced matrix (*Table 7·61*)

Table 7·61

L_2				
0	3	6	9	
0	1	2	3	
0	1	2	3	
0	0	0	0	$\rightarrow L_1$

Since all zeros in *Table 7·61* can be covered by minimum number of lines (L_1, L_2), which is less than 4 (the number of rows in the matrix), the optimal assignment is not possible at this stage.

Step 5. In *Table 7·61*, select the minimum element '1' among all uncovered elements. Then subtract this value 1 from each uncovered element, and add 1 at the intersection of two lines L_1, L_2. Thus, the revised matrix is obtained as *Table 7·62*.

Table 7.62

L_2	L_3			
0	2	5	8	
0	0	1	2	
0	0	1	2	
0	0	0	0	$\rightarrow L_1$

Step 6. Again, repeat *Step 5*. Since the minimum number of lines (L_1, L_2, L_3) in *Table 14·62* to cover all zeros is less than 4 (the number of rows/columns), subtract the min. element 1 from all uncovered elements and add 1 at the intersection of lines (L_1, L_2) and (L_1, L_3). Then find the optimal assignment as explained in *Step 7.*

Step 7. To find an optimal assignment.

Since there is a single zero element in row 1 and column 4 only, make the zero assignment by putting '□' around these two zeros and cross-out other zeros in column 1 and row 4. Other zero-assignments are quite obvious from the following tables :

Table 7·63 (a)

	I	II	III	IV
A	$\boxed{0}$	2	4	7
B	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	1
C	$\cancel{0}$	$\cancel{0}$	$\boxed{0}$	1
D	2	1	$\cancel{0}$	$\boxed{0}$

Table 7·63 (b)

	I	II	III	IV
A	$\boxed{0}$	2	4	7
B	$\cancel{0}$	$\cancel{0}$	$\boxed{0}$	1
C	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	1
D	2	1	$\cancel{0}$	$\boxed{0}$

Thus, two possible solutions are : *(i)* A-I, B-II, C-III, D-IV ; *(ii)* A-I, B-III, C-II, D-IV.

Both the solutions show that the best salesman *A* is assigned to the richest territory I, the worst salesman *D* to the poorest territory IV. Salesman *B* and *C* being equally good, so they may be assigned to either II or III. This verifies the answer.

Example 16. *A manufacturing company has four zones A, B, C, D and four sales engineers P, Q, R, S respectively for assignment. Since the zones are not equally rich in sales potential, it is estimated that a particular engineer operating in a particular zone will bring the following sales :*

Zone A	:	4,20,000
Zone B	:	3,36,000
Zone C	:	2,94,000
Zone D	:	4,62,000

The engineers are having different sales ability. Working under the same conditions, their yearly sales are proportional to 14, 9, 11 and 8 respectivley. The criteria of maximum expected total sales is to be met by assigning the best engineer to the richest zone, the next best to the second richest zone and so on.

Find the optimum assignment and the maximum sales. **(C.A. (May) 98)**

Solution. ***Step 1 : Construct the Effectiveness Matrix.*** To avoid the fractional values of annual sales of each sales engineer in each zone, for convenience consider their yearly sales as 42 (*i.e.*, the sum of sales proportions), taking Rs. 1000 as one unit. Now divide the individual sales in each zone by 42 to obtain the required annual sales by each sales-engineer. The maximum sales matrix so obtained is given in ***Table 7.64***

Table 7.64 : Effectiveness Matrix

Sales Engineer \ Zones	A	B	C	D	Sales proportion
P	140	112	98	154	14
Q	90	72	63	99	9
R	110	88	77	121	11
S	80	64	56	88	8
Sales (in Rs. 1000)	10	8	7	11	

Step 2 **: Converting Maximization Problem into Minimization** : The given maximization assignment problem ***(Table 7.64)*** can be converted into a minimization assignment problem by subtracting from the highest element (*i.e.*, 154), all the elements of the given table. The resulting matrix so obtained is given in ***Table 7.65*** below :

Step 3 : Subtracting the smallest element in each row from every element in that row and smallest element in each column from every element in that column, we get the following reduced matrix ***(Table 7.66)***. Since all zeros in ***Table 7.65*** can be covered by minimum number of lines (L_1, L_2) which is less than the number of rows (4) in the matrix, the optimum assignment is not possible at this stage and we pass to the next step.

Table 7.65 : Equivalent Cost Table

Sales Engr. \ Zone	A	B	C	D
P	14	42	56	0
Q	64	82	91	55
R	44	66	77	33
S	74	90	98	66

Table 7.66

Sales Engr. \ Zone	A	B	C	D
P	6	18	24	0
Q	1	3	4	0
R	3	9	12	0
S	0	0	0	0

Step 4. Select the minimum element '1' among all uncovered elements and subtract this value from each uncovered element and add '1' at the intersection of two lines L_1, L_2. Draw more minimum possible number of lines so as to cover the new zeros ***(Table 7.67)***.

Table 7.67

Sales Engr. \ Zone	A	B	C	D
P	5	17	23	0
Q	0	2	3	0
R	2	8	11	0
S	0	0	0	0

Step 5. Since the number of lines is still less than the order of the sale matrix, we repeat the procedure and obtain ***Table 7.68*** and ***Table 7.69.*** The table shows that the number of lines is equal to order of the sale matrix and hence an optimum solution is obtained.

Table 7.68

Zone → / Sales Engr. ↓	A	B	C	D	
P	5	15	21	0	
Q	0	0	1	0	→ L_2
R	2	6	9	0	
S	0	0	0	3	→ L_3
				↓ L_1	

Table 7.69

Zone → / Sales Engr. ↓	A	B	C	D	
P	3	13	19	0	
Q	0	0	1	0	→ L_2
R	0	4	7	0	
S	3	0	0	3	→ L_3
	↓ L_4			↓ L_1	

Step 6. To determine the optimum assignment, we first observe that there is only single zero element in row 1 and column 3, so we make the zero assignment by putting '☐' around these two zeros and cross-out other zeros in column 4 and row 4. The remaining zero-assignments are quite obvious from the adjoining ***Table 7.70***.

Table 7.70

Zone → / Sales Engr. ↓	A	B	C	D
P	3	13	19	$\boxed{0}$
Q	$\cancel{0}$	$\boxed{0}$	1	2
R	$\boxed{0}$	4	7	$\cancel{0}$
S	2	$\cancel{0}$	$\boxed{0}$	5

The optimum solution is obtained and the assignment is :

$$P \to D, Q \to B, R \to A \text{ and } S \to C$$

The solution shows that the best salesman *P* is assigned to the richest zone *D* and the worse salesman *S* to the poorest zone *C*. The second best salesman to the next richest zone *A* and so on.

$$\text{Maximum sales} = \text{Rs. } 154 + 72 + 110 + 56 = \text{Rs. } 392 \text{ thousands}$$
$$= \text{Rs. } 392000.$$

Example 17. *Five salesmen are to be assigned to five territories. Based on the past performance, the following table shows the annual sales (in rupees lakhs) that can be generated by each salesman in each territory. Find the optimum assignment.*

Territory / Salesman	T_1	T_2	T_3	T_4	T_5
S_1	26	14	10	12	9
S_2	31	27	30	14	16
S_3	15	18	16	25	30
S_4	17	12	21	30	25
S_5	20	19	25	16	10

[A.I.M.A. (P.G. (Dip. in Management)), Dec. 96]

Solution. ***Step 1.*** Since the matrix represents the sales which can be generated by each territory, the objective function of the assignment problem is, therefore, to maximize the total sales generated. But the algorithm for assignment problem is for minimization of the objective function, We, therefore, convert the given problem to minimization problem, by subtracting all the elements of the given matrix from the maximum element 31 to obtain the adjoining matrix.

Territory / Salesman	T_1	T_2	T_3	T_4	T_5
S_1	5	17	21	19	22
S_2	0	4	1	17	15
S_3	16	13	15	6	1
S_4	14	19	10	1	6
S_5	11	12	6	15	21

Step 2. Row subtraction

Territory / Salesman	T_1	T_2	T_3	T_4	T_5
S_1	0	12	16	14	17
S_2	0	4	1	17	15
S_3	15	12	14	5	0
S_4	13	18	9	0	5
S_5	5	6	0	9	15

Column subtraction

Territory / Salesman	T_1	T_2	T_3	T_4	T_5
S_1	0	8	16	14	17
S_2	0	0	1	17	15
S_3	15	8	14	5	0
S_4	13	14	9	0	5
S_5	5	2	0	9	15

Step 3. Minimum straight lines to cover zeros.

Territory→ Salesman ↓	T_1	T_2	T_3	T_4	T_5	
S_1	0	12	16	14	17	
S_2	0	4	1	17	15	L_2
S_3	15	12	14	5	0	
S_4	13	18	9	0	5	
S_5	5	6	0	9	15	
	L_1		L_3	L_4	L_5	

Step 4. Since number of lines is 5, the optimality criteria is satisfied.

Territory→ Salesman ↓	T_1	T_2	T_3	T_4	T_5
S_1	$\boxed{0}$	8	16	14	17
S_2	$\cancel{0}$	$\boxed{0}$	1	17	15
S_3	15	8	14	5	$\boxed{0}$
S_4	13	14	9	$\boxed{0}$	5
S_5	5	2	$\boxed{0}$	9	15

The optimum assignment is :

$$S_1 \to T_1,\ S_2 \to T_2,\ S_3 \to T_5,\ S_4 \to T_4,\ S_5 \to T_3$$

and the maximum sales generated are :

$$26 + 27 + 30 + 30 + 25 = 138.$$

EXAMINATION PROBLEMS

1. In an assignment problem, there are 12 workers and 12 jobs to be done. Only one man can work on any one job. What is the total number of different possible ways of assignment if the jobs to the worker 3 ? **[IGNOU 99 (Dec.)]**
   [**Ans.** 12 ! = 479001600 ways]

2. A marketing manager has 5 salesman and 5 sales districts. Considering the capabilities of the salesman and the nature of districts, the marketing manager estimates that sales per month (in hundred rupees) for each salesman in each district would be as follows :

	A	B	C	D	E
1	32	38	40	28	40
2	40	24	28	21	36
3	41	27	33	30	37
4	22	38	41	36	36
5	29	33	40	35	39

   Find the assignment of salesman to districts that will result in a maximum sale.
   [Agra 98; Rohil. 90; Bharathidasan B.Sc. 90]
   [**Ans.** 1 → B , 2 → A, 3 → E , 4 → C and 5 → D , max. profit = Rs. 191]

3. The owner of a small mechine shop has four machinists available to do jobs for the day. Five jobs are offered with expected profit for each machinist on each job as follows :

	1	2	3	4
A	32	41	57	18
B	48	54	62	34
C	20	31	81	57
D	71	43	41	47
E	52	29	51	50

   Find by using the assignment method, the assignment of machinists to jobs that will result in a maximum profit. Which job should be declined. **[JNTU (Mech) 99, 98]**
   [**Ans.** A → dummy, B → 2, C → 3, D → 1, E → 4 ; max profit = 256]

4. A company has six jobs to be processed by six mechanics. The following table gives the return in rupees when the ith job is assigned to the jth mechanic ($i, j = 1, \ldots, 6$) . How should the jobs be assigned to the mechanics so as to maximize the overall return.

Job

		I	II	III	IV	V	VI
	1	9	22	58	11	19	27
	2	43	78	72	50	63	48
Mechanic	3	41	28	91	37	45	33
	4	74	42	27	49	39	32
	5	36	11	57	22	25	18
	6	3	56	53	31	17	28

[Meerut B.Sc. (Math.) 90 ; M.Sc. Baroda B.Sc. (Math.) 81]
[Ans. 1 → VI , 2 → V , 2 → III , 4 → I , 5 → IV , 6 → II , max. return = 333 .

5. Five lathes are to be allotted to five operators (one for each). The following table gives weekly output figures (in pieces) :

Weekly Output in Lathe

		L_1	L_2	L_3	L_4	L_5
	P	20	22	27	32	36
	Q	19	23	29	34	40
Operator	R	23	28	35	39	34
	S	21	24	31	37	42
	T	24	28	31	36	41

Profit per piece is Rs. 25. Find the maximum profit. [C.A. (Nov.) 93]

[Hint : The given problem is of maximization one. So convert it into an opportunity loss matrix by subtracting all the elements from the highest element 42.]

[Ans. : $P \to L_1$, $Q \to L_5$, $R \to L_3$, $S \to L_4$, $T \to L_2$. Max. weekly output = 160 pieces.
Maximum profit = 25 × 160 = Rs. 4000.]

7·7-2 Restrictions on Assignment

Sometimes technical, legal or other restrictions do not permit the assignment of a particular facility to a particular job. Such difficulty can be overcome by assigning a very high cost (say, infinite cost) to the corresponding cell, so that the activity will be automatically excluded from the optimal solution. The following example will make the procedure clear.

Q. How will you solve an assignment problem, where a particular assignment is prohibited.

Example 18. *A job shop has purchased 5 new machines of different type. There are 5 available locations in the shop where a machine could be installed. Some of these locations are more desirable than others for particular machines because of their proximity to work centres which would have a heavy work flow to and from these machines. Therefore, the objective is to assign the new machines to the available locations in order to minimize the total cost of material handling. The estimated cost per unit time of materials handling involving each of the machines is given below for the respective locations. Locations 1, 2, 3, 4 and 5 are not considered suitable for machines A, B, C, D and E, respectively. Find the optimal solution :*

Location (Cost in Rs.)

		1	2	3	4	5
	A	×	10	25	25	10
	B	1	×	10	15	2
Machine	C	8	9	×	20	10
	D	14	10	24	×	15
	E	10	8	25	27	×

How would the optimal solution get modified if location 5 is also unsuitable for machine A ?

Solution. Since locations 1, 2, 3, 4 and 5 are not suitable for machines A, B, C, D and E respectively, an extremely large cost (say ∞) should be attached to these locations. Then the cost matrix of the resulting assignment problem becomes as shown below :

∞	10	25	25	10
1	8	10	15	2
8	9	∞	20	10
14	10	24	∞	15
10	8	25	27	∞

∞	2	6	3	$\boxed{0}$
$\cancel{0}$	∞	$\boxed{0}$	2	1
$\cancel{0}$	3	∞	$\boxed{0}$	2
2	$\boxed{0}$	3	∞	3
$\boxed{0}$	$\cancel{0}$	6	5	∞

Following the usual procedure of solving an assignment problem, the optimum assignment is obtained as shown in the above right side table. This gives the optimal assignment as:

$A \to 5$, $B \to 3$, $C \to 4$, $D \to 2$, and $E \to 1$, with total min cost = Rs. 60 .

Now if location 5 is also not suitable for the machine A, we attach an extremely large cost (= ∞) to cell (1, 5). Again applying the assignment procedure to this modified problem, the following assignment solution can be easily obtained.

$$\left.\begin{array}{l} A \to 4, B \to 3, C \to 5, D \to 2, \text{ and } E \to 1 \\ \text{or} \quad A \to 2, B \to 3, C \to 4, D \to 5, \text{ and } E \to 1 \end{array}\right\} \text{ with min cost of Rs. 65.}$$

Q. Explain how to modify an effectiveness matrix in an assignment problem if a particular assignment is prohibited.

EXAMINATION PROBLEMS

1. Five operators have to be assigned to five machines. The assignment costs are given in the table below:

Machine

Operator		I	II	III	IV	V
	A	5	5	—	2	6
	B	7	4	2	3	4
	C	9	3	5	—	3
	D	7	2	6	7	2
	E	6	5	7	9	1

Operator A cannot operate machine III and operator C cannot operate machine IV. Find the optimal assignment schedule.

[Ans. A → IV , B → III , C → II , D → I , E → V ; or A → IV , B → III , C → V , D → II , E → I , min cost = 15 .]

2. Four operators O_1, O_2, O_3 and O_4 are available to a manager who has to get four jobs J_1, J_2, J_3 and J_4 done by assigning one job to each operator. Given the times needed by different operators for different jobs in the matrix below:

	J_1	J_2	J_3	J_4
O_1	12	10	10	8
O_2	14	12	15	11
O_3	6	10	16	4
O_4	8	10	9	7

(i) How should the manager assign the jobs so that the total time needed for all four jobs is minimum ?

(ii) If job J_2 is not to be assigned to operator O_2, what should be the assignment and how much additional total time will be required? **[C.A. (May) 94]**

[Ans. (i) $O_1 \to J_3$, $O_2 \to J_2$, $O_3 \to J_4$, $O_4 \to J_1$, min. time = 34. (ii) $O_1 \to J_2$, $O_2 \to J_4$, $O_3 \to J_1$, $O_4 \to J_3$, min time = 36. } ∴ Additional time required = 36 − 34 = 2 units of time.

3. Five swimmers are eligible to complete in a relay team which is to consist of four swimmers swimming four different swimming styles; back stroke, breast stroke, free style and butterfly. The time taken for the five swimmers—Anand Bhasker, Chandru, Dorai and Easwar—to cover a distance of 100 meters in various swimming styles are given below in minutes, seconds.

Anand swims the back stroke in 1 : 09, the breast stroke in 1 : 15 and has never competed in the free style or butterfly.

Bhasker is a free style specialist averaging 1 : 01 for the 100 meters but can also swim the breast stroke in 1 : 16 and butterfly in 1 : 20.

Chandru swims all styles—back 1 : 10, butterfly 1 : 12, free style 1 : 05 and breast stroke 1 : 20.

Dorai swims only the butterfly 1 : 11 while Easwar swims the back stroke 1 : 20, the breast stroke 1 : 16, the free style 1 : 06 and the butterfly 1 : 10.

Which swimmer should be assigned to which swimming style ? Who will not be in the relay. **[C.A. (Nov.) 91]**

[Hint. The assignment matrix with time expressed in seconds and adding a dummy style to balance it is given by

	Back stroke	Breast stroke	Free style	Butterfly	Dummy
Anand	69	75	–	–	0
Bhasker	–	76	61	80	0
Chandru	70	80	65	72	0
Dorai	–	–	–	71	0
Easwar	80	76	66	70	0

[Ans. Anand will be in Breast stroke, (time 75 secs.)
Bhasker will be in free stroke, (time 61 secs.)
Chandru will be in Back stroke, (time 70 secs.)
Dorai will not participate (dummy), (time 70 secs.)
Easwar will be in Butterfly, (time 70 secs.)
Dorai will be out of the relay.
Total minimum time in the relay (= 276 secs or 4 min. 36 sec.)

4. WELLDONE company has taken the third floor of a multistoreyed building for rent with a view to locate one of their zonal offices. There are five main rooms in this floor to be assigned to five managers. Each room has its own advantages and disadvantages. Some have windows, some are closer to washrooms or to the canteen or secretarial pool. The rooms are of all different sizes and shapes. Each of the five managers were asked to rank their room preferences amongst the rooms 301, 302, 303, 304 and 305. Their preferences were recorded in a table as indicated below.

M_1	M_2	M_3	M_4	M_5
302	302	303	302	301
303	304	301	305	302
304	305	304	304	304
	301	305	303	
		302		

Most of the managers did not list all the five rooms since they were not satisfied with some of these rooms and they have left off these from the list. Assuming that their preferences can be quantified by numbers, find out as to which manager should be assigned to which room so that their total preference ranking is a minimum. [C.A. (Nov.) 90]

[Hint. Formulation of preference ranking assignment problem is :

		M_1	M_2	M_3	M_4	M_5
	301	–	4	2	–	1
	302	1	1	5	1	2
Room No.	303	2	–	1	4	–
	304	3	2	3	3	3
	305	–	3	4	2	–

[Ans. $M_1 \rightarrow 302$, $M_2 \rightarrow 304$, $M_3 \rightarrow 303$, $M_4 \rightarrow 305$, $M_5 \rightarrow 301$, and the total minimum ranking is $1 + 2 + 1 + 2 + 1 = 7$?

5. Imagine yourself to be the *Executive Director* of a 5-star Hotel which has four banquet halls that can be used for all functions including weddings. The halls were all about the same size and the facilities in each hall differed. During a heavy marriage seen, 4 parties approached you to reserve a hall for the marriage to be celebrated on the same day. These marriage parties were told that the first choice among these 4 halls would cost Rs. 10,000 for the day. They were also required to indicate the second, third and fourth preferences and the price that they would be willing to pay marriage party A & D indicated that they won't be interested in Halls 3 & 4. Other particulars are given in the following table :

Revenue/Hall Table

Marriage Party	Hall			
	1	2	3	4
A	10,000	9,000	×	×
B	8,000	10,000	8,000	5,000
C	7,000	10,000	6,000	8,000
D	10,000	8,000	×	×

Where × indicates that the party does not want that Hall. Decide on an allocation that will maximize the revenue to your Hotel. [C.A. (May) 95]

[Hint. To solve this problem of maximization, first convert it to a minimization problem by subtracting all the elements of the given matrix from its highest element which is equal to Rs. 10,000 here. The matrix thus obtained will be named as loss matrix. Now apply assignment algorithm to the loss matrix.]

[Ans. Marriage party A $\rightarrow$ Hall 2, B $\rightarrow$ 3, C $\rightarrow$ 4 and D $\rightarrow$ 1.
Maximum revenue = Rs. (9,000 + 8,000 + 8,000 + 10,000) = Rs. 35,000.]

6. The secretary of a School is taking bids on the city's four school bus routes. Four companies have made the bids as detailed in the following table :

Company	Bids (in Rs.)			
	Route 1	Route 2	Route 3	Route 4
1	4,000	5,000	–	–
2	–	4,000	–	4,000
3	3,000	–	2,000	–
4	–	–	4,000	5,000

Suppose each bidder can be assigned only one route. Use the assignment model to minimize the school's cost of running the four bus routes. [C.A. (Nov.) 95]

[**Hint.** Since some of the companies have not made bids for certain routes, assign a very high bid M for all such routes. Then apply assignment algorithm.]

[**Ans.** Company 1 → Route 1, Company 2 → Route 2, Company 3 → Route 3, Company 4 → Route 4. Minimum cost = Rs. (4000 + 4000 + 2000 + 5000) = Rs. 15,000.]

7. Suggest optimum assignment of 4 workers *A, B, C* and *D* to 4 jobs, I, II, III and IV. The time taken by different workers in completing the different jobs is given below :

		Job			
		I	II	III	IV
Worker	A	8	10	12	16
	B	11	11	15	8
	C	9	6	5	14
	D	15	14	9	7

Also indicate the total time taken in completing the jobs. **[Raj. (M. Com.) 98; Delhi (M. Com.) 96]**

[**Ans.** Optimum assignment is :

$$I \xrightarrow{8} A,\ II \xrightarrow{11} B,\ III \xrightarrow{5} C,\ IV \xrightarrow{7} D,\ \text{Min cost} = 31\ ]$$

8. The XYZ company has 5 jobs *I, II, III, IV V* to be done and 5 men *A, B, C, D, E* to do these jobs. The number of hours each man would take to accomplish each job is given by the following table.

	L	*M*	*N*	*O*	*P*
A	16	13	17	19	20
B	14	12	13	16	17
C	14	11	12	17	18
D	5	5	8	8	11
E	5	3	8	8	10

Work out the optimum assignment and the total minimum time taken. **[Allahabad (M.B.A.) Feb., 99]**

[**Ans.** $A \xrightarrow{13} M,\ B \xrightarrow{17} P,\ C \xrightarrow{12} N,\ D \xrightarrow{5} L,\ E \xrightarrow{8} O$ cost = 55]

9. Given the following data, determine the least cost allocation of the available machines to five jobs.

		Job				
		A	*B*	*C*	*D*	*E*
Machine	1	25	29	31	42	37
	2	22	19	35	18	26
	3	39	38	26	20	33
	4	34	27	28	40	32
	5	24	42	36	23	45

[Andhra (M.B.A.) 98]

[**Ans.** $1 \xrightarrow{25} A,\ 2 \xrightarrow{19} B,\ 3 \xrightarrow{26} C,\ 4 \xrightarrow{32} E,\ 5 \xrightarrow{25} D$ cost = 125]

10. A company receives tenders for four projects from five contractors. Only one project can be assigned to any contractor. The tender received (in thousands of rupees) are given below. Contractor *D* does not want to carry out project 2 and has not, therefore, submitted the tender for that.

		Contractor				
		A	*B*	*C*	*D*	*E*
Project	1	500	600	150	450	600
	2	400	550	200	—	550
	3	450	575	175	425	610
	4	475	575	185	440	590

(i) Using the Hungarian method, find the set of assignments with the smallest possible total cost.

(ii) What will be the minimum cost of getting all the projects completed ? **[Delhi (M. Com.) 98]**

11. The Secretary of a School is taking bids on the City's four school bus routes. Four companies have made the bids as detailed in the following table :

	BIDS			
	Route 1	Route 2	Route 3	Route 4
Company 1	Rs. 4,000	Rs. 5,000	—	—
Company 2	—	Rs. 4,000	—	Rs. 4,000
Company 3	Rs. 3,000	—	Rs. 2,000	—
Company 4	—	—	Rs. 4,000	Rs. 5,000

Suppose each bidder can be assigned only one route. Use the assignment model to minimize the school's cost of running the four bus routes. [C.A. (Nov.) 95]

[Hint.

Steps 1 : Row reduction

	BIDS			
	Route 1	Route 2	Route 3	Route 4
Company 1	$\boxed{0}$	1,000	—	—
Company 2	—	$\boxed{0}$	—	0
Company 3	1,000	—	$\boxed{0}$	—
Company 4	—	—	0	1,000

Steps 2 : Since each column has a zero, the column reduction will give the same matrix. Let us start assigning as usual.

	Route 1	Route 2	Route 3	Route 4
Company 1	0	1,000	—	—
Company 2	—	0	—	0
Company 3	1,000	—	0	—
Company 4	—	—	0	1,000

The minimum uncovered element is 1,000.

Step 3. Performing the usual operations, Step 4 : The desired solution is

	Route 1	Route 2	Route 3	Route 4
Company 1	$\boxed{0}$	1,000	—	—
Company 2	—	$\boxed{0}$	—	0
Company 3	1,000	—	$\boxed{0}$	—
Company 4	—	—	0	$\boxed{0}$

Company	Route	Cost (Rs.)
1	1	4,000
2	2	4,000
3	3	2,000
4	4	5,000

12. An organisation producing 4 different products, *viz.*, *A, B, C* and *D* having 4 operators, *viz.*, *P, Q, R* and *S* who are capable of producing any of the four products, works effectively 7 hours a day. The time (in minutes) required for each operator for producing each of the products are given in the cells of the following matrix along with profit (Rs. per unit) :

	Product			
Operator	A	B	C	D
P	6	10	14	12
Q	7	5	3	4
R	6	7	10	10
S	20	10	15	15
Profit (Rs./unit)	3	2	4	1

Find out the assignment of operators to product which will maximize the profit. [C.A. (May) 96]

[Hint. Step 1 : Given that the unit (factory) works effectively for 7 hours and the processing time (in minutes) for each of the four products by different operators, we obtain the production and profit matrices as follows :

Production Matrix

Operator	Product			
	A	B	C	D
P	70	42	30	35
Q	60	84	140	15
R	70	60	42	42
S	20	42	28	28

Profit Matrix

Operator	Product			
	A	B	C	D
P	210	84	120	35
Q	180	168	560	105
R	210	120	168	42
S	60	84	112	28

To use the same algorithm for minimization, subtract all the elements from the highest value and obtain the following matrix :

Operator	Product			
	A	B	C	D
P	350	476	440	525
Q	380	392	0	455
R	350	440	392	518
S	500	476	448	532

Subtracting row minimum, we obtain

Operator	Product			
	A	B	C	D
P	0	126	90	175
Q	380	392	0	455
R	0	90	42	168
S	52	28	0	84

Since required number of assignments could not be made, proceed further.

Operator	Product			
	A	B	C	D
P	$\boxed{0}$	36	90	29
Q	380	302	$\boxed{0}$	309
R	0	$\boxed{0}$	42	22
S	114	0	62	$\boxed{0}$

Subtracting column minimum and after assignment, we get

Operator	Product			
	A	B	C	D
P	$\boxed{0}$	98	90	91
Q	380	364	$\boxed{0}$	371
R	0	62	42	84
S	52	$\boxed{0}$	0	—

The optimum solution is :

Operator	Product	Profit (Rs.)
P	A	210
Q	C	560
R	B	120
S	D	28
Total profit		918

13. (a) A firm produces four products. There are four operators who are capable of producing any of these four products. The processing time varies from operator to operator. The firm records 8 hours a day and allows 30 minutes for lunch. The processing time in minutes and the profit for each of the product are given below :

Operator	Product			
	A	B	C	D
1	15	9	10	6
2	10	6	9	6
3	25	15	15	9
4	15	9	10	10
Profit (Rs. per unit)	8	6	5	4

Find the optimum assignment of product to operators. **[C.A. Nov., 97]**

(b) Five lathes are to be allotted to five operators (one for each). The following table gives weekly output figures (in pieces) :

	Lathes				
Operator	L_1	L_2	L_3	L_4	L_5
P	20	22	27	32	36
Q	19	23	29	34	40
R	23	28	35	39	34
S	21	24	31	37	42
T	24	28	31	36	41

Profit per piece is Rs. 25. Find the maximum profit per week.

[**Hint.** The net working time is 450 minutes per day. The number of items that could be produced by the four operators is given below :

Operator	Product			
	A	B	C	D
1	30	50	45	75
2	45	75	50	75
3	18	30	30	50
4	30	50	45	45

Multiplying with the corresponding profit, we obtain the following matrix for finding maximizing profit.

Operator	Product			
	A	B	C	D
1	240	300	225	300
2	360	450	250	300
3	144	180	150	200
4	240	300	225	180

FINAL TABLE : ***Optimum solution***

Operator	Product			
	A	B	C	D
1	0	0	21	$\boxed{0}$
2	30	$\boxed{0}$	146	150
3	0	24	$\boxed{0}$	4
4	$\boxed{0}$	0	21	120

The assignment is

Operator		Product	Profit
1	→	D	300
2	→	B	450
3	→	C	150
4	→	A	240
		Rs.	1,140

14. The marketing director of a multi-unit company is faced with a problem of assigning 5 senior managers to six zones. From the past experience he knows that the efficiency percentage judged by sales, operating cost etc. depends on manager-zone combination. the efficiency of different managers is given below :

Manager	Zone I	II	III	IV	V	VI
A	73	91	87	82	78	80
B	81	85	69	76	74	85
C	75	72	83	84	78	91
D	93	96	86	91	83	82
E	90	91	79	89	69	76

Find out which zone will be managed by a junior manager due to non-availability of senior manager.

[Poona (M.B.A.) 98]

[**Ans.** *A* → *III*, *B* → *II*, *C* → *VI*, *D* → *I*, *E* → *IV*, *F* → *V* and zone V should be managed by a junior manager.]

15. To stimulate interest and provide an atmosphere for intellectual discussion, a finance faculty in a management school decides to hold special seminars to four contemporary topics—leasing, portfolio management, private mutual funds, swaps and options. Such seminars should be held once in a week in the afternoons. However, scheduling these seminars (one for each topic, and not more than one seminar per afternoon) has to be done carefully so that the number of students unable to attend is kept to a minimum. A cereful study indicates that the number of students who cannot attend a particular seminar on a specific day is as follows :

	Leasing	Portfolio management	Private mutual funds	Swaps and options
Monday	50	40	60	20
Tuesday	40	30	40	30
Wednesday	60	20	30	20
Thursday	30	30	20	30
Friday	10	20	10	30

Find an optimum schedule of the seminars. Also find out the total number of students who will be missing at least one seminar. **[C.A., May 99]**

[**Hint.** Final Table : Optimum Solution

		I	II	III	IV	V
		Leasing	Portfolio Management	Private Mutual Funds	Swaps and options	Dummy
1.	Mon.	30	20	40	0	0
2.	Tues.	20	10	20	10	10
3.	Wed.	40	0	10	0	0
4.	Thurs.	10	10	0	10	0
5.	Fri.	0	10	0	20	10

[**Ans.** $1 \xrightarrow{20} \text{IV}$, $2 \xrightarrow{0} \text{V}$, $3 \xrightarrow{20} \text{II}$, $4 \xrightarrow{20} \text{III}$, $4 \xrightarrow{10} \text{I}$, cost = 70.]

16. *XYZ* Airline operating 7 days a week has given the following time-table. Crews must have a minimum layover of 5 hours between flights. Obtain the pairing flights that minimizes layover time away from home. For any given pairing the crew will be based at the city that results in the smaller layover :

Chennai—Mumbai			Mumbai—Chennai		
Flight Number	Depart.	Arrive.	Flight Number	Depart.	Arrive.
A_1	6 AM	8 AM	B_1	8 AM	10 AM
A_2	8 AM	10 AM	B_2	9 AM	11 AM
A_3	2 PM	4 PM	B_3	2 PM	4 PM
A_4	8 PM	10 PM	B_4	7 PM	9 PM

[Hint. See Example 5.]

[Ans. $A_1 \to B_3$, $A_2 \to B_4$, $A_3 \to B_1$, $A_4 \to B_2$ *, Min. Layover time 40 hrs.] **[CA (May) 2000]**

17. Due to absence of a workman, an officer has to assign four out of five different jobs to four workers with the performance matrix given below :

		Operators			
		A	B	C	D
Jobs	1	3	6	5	3
	2	4	9	3	2
	3	11	2	4	6
	4	10	4	6	5
	5	11	12	14	10

[Ans. 1 → *A*, 2 → *C*, 3 → *B*, 4 → *D*, 5 – *D'* or 1 → *A*, 2 → *D*, 3 → *C*, 4 → *B*, 5 – *D'*] **[IGNOU 2000]**

18. Four different jobs can be done on four different machines. The set up and down time costs are prohibitively high for change overs. The matrix below gives the cost in hundreds of Rs. for job J_i to M_j.

		Machine			
		M_1	M_2	M_3	M_4
Job	J_1	10	13	9	15
	J_2	12	10	12	9
	J_3	16	14	15	13
	J_4	11	11	12	8

Assign the jobs to the machines in order to minimize the total cost. **[AIMS (Banglore) MBA 2002]**

7.8 SENSITIVITY IN ASSIGNMENT PROBLEMS

The structure of assignment problem is of such a type that there is very little scope for sensitivity analysis. Modest alterations in the conditions (such as one being able to do two jobs) can be considered by repeating the man's row and adding a dummy column to square up the matrix.

Addition of a constant throughout any row or column also makes no difference to this position of optimal assignment. However, sometimes equiproportionate change throughout a row or column can make a difference. So in reference to assignment problems there is no scope for altering the level of an assignment.

Q. 1. (a) What is Assignment Problem ? Give two areas of its applications.

(b) How far sensitivity analysis is relevent to assignment.

2. Show that the lines drawn in the assignment algorithm pass through all the zeros and have the property that every line passes through one and only one assignment.

3. Show that in an assignment, if we multiply each element of the effectiveness (cost) matrix by some fixed number, then the optimal solution remains unchanged.

EXAMINATIONS REVIEW PROBLEMS

1. Alpha corporation has four plants each of which can manufacture any one of four products. Production costs differ from one plant to another as do sales revenue. Given the revenue and cost data below, obtain which product each plant should produce to maximize profit.

Sales revenue (Rs.' 000s) Product

Plant	1	2	3	4
A	50	68	49	62
B	60	70	51	74
C	55	67	53	70
D	58	65	54	69

Production costs (Rs.' 000s)

Plant	1	2	3	4
A	49	60	45	61
B	55	63	45	69
C	52	62	49	68
D	55	64	48	66

[**Hint.** Construct the profit matrix by using the fact : Profit matrix = Revenue matrix – Cost matrix.
To make use of minimization technique, subtract each element of profit matrix from the maximum element which will be 8. Then apply assignment rule in usual manner.]
[**Ans.** A – 2, B – 4, C – 1, D – 3.]

2. Find the optimal solution for the assignment problem with the following cost matrix.

(i)

	I	II	III	IV	V
A	11	17	8	16	20
B	9	7	12	6	15
C	13	16	15	12	16
D	21	24	17	28	26
E	14	10	12	11	15

(ii)

	I	II	III	IV
A	5	3	1	8
B	7	9	2	6
C	6	4	5	7
D	5	7	7	6

[I.A.S. (Main) 2000]

[**Ans.** (i) A – I, B – IV, C – V, D – III, E – II, min cost = 60, (ii) A–III, B–IV, C–II, D–I, min cost = 16]

3. Solve the following assignment problems :

(i) Men

Tasks	I	II	III	IV	V
A	1	3	2	8	8
B	2	4	3	1	5
C	5	6	3	4	6
D	3	1	4	2	2
E	1	5	6	5	4

[**Ans.** A – I, B – IV, C – III, D – II, E – V].

(ii) Persons

Tasks	1	2	3	4
A	10	12	19	11
B	5	10	7	8
C	12	14	13	11
D	8	15	11	9

[**Ans.** A $\to$ 2, B $\to$ 3, C $\to$ 4, D $\to$ 1 min cost = 38].

(iii)

	1	2	3	4
I	12	30	21	15
II	18	33	9	31
III	44	25	24	21
IV	23	30	28	14

[**Ans.** I – 1, II – 3, III – 2, IV – 4]

(iv)

	I	II	III	IV
1	2	3	4	5
2	4	5	6	7
3	7	8	9	8
4	3	5	8	4

[Meerut (Maths) 91]

[**Ans.** (i) 1 – I, 2 – II, 3 – III,
(ii) 1 – II, 2 – I, 3 – IV, 4 – III : and others]

(v) Man

Jobs	1	2	3	4	5
I	12	8	7	15	4
II	7	9	17	14	10
III	9	6	12	6	7
IV	7	6	14	6	10
V	9	6	12	10	6

[**Ans.** (i) I – 3, II – 1, III – 2, IV – 4, V – 5 ;
(ii) I – 3, II – 1, III – 4, IV – 2, V – 5].

(vi) Job

Person	1	2	3	4	5
A	8	4	2	6	1
B	0	9	5	5	4
C	3	8	9	2	6
D	4	3	1	0	3
E	9	5	8	9	5

[IAS (Maths) 89]

[**Ans.** A $\to$ 5, B $\to$ 1, C $\to$ 4, D $\to$ 3, E $\to$ 2, min. cost = 9]

(vii)

	I	II	III	IV	V
A	15	21	17	4	9
B	3	40	21	10	7
C	9	6	5	8	10
D	14	8	6	9	3
E	21	16	18	7	4

[**Ans.** A – V, B – I, C – II, D – III, E – V].

(viii)

	I	II	III	IV	V
A	6	5	8	11	16
B	1	13	16	1	10
C	16	11	8	8	8
D	9	14	12	10	16
E	10	13	11	8	16

[**Ans.** (i) A $\to$ II, B $\to$ I, C $\to$ V, D $\to$ III, E $\to$ IV;
(ii) A $\to$ II, B $\to$ IV, C $\to$ V, D $\to$ 1, E $\to$ III, min cost = 34].

4. There are five jobs to be assigned one each to 5 machines and the associated cost matrix is as follows :

Machine

		1	2	3	4	5
	A	11	17	8	16	20
	B	9	7	12	6	15
Job	C	13	16	15	12	16
	D	21	24	17	28	26
	E	14	10	12	11	15

[**Ans.** $A-1$, $B-4$, $C-5$, $D-3$, $E-2$ min cost = Rs. 60].

5. An air freight company picks up and delivers freight where customers require. Company has two types of aircrafts X and Y with equal loading capacities but different operating costs. These are shown below :

Type of aircraft	Cost per freight	Freight (Rs.)
	Empty	Loaded
X	1·00	2·00
Y	1·50	3·00

The present four locations of the aircrafts which the company is having are as shown below :

$$J \to X,\ K \to Y,\ L \to Y;\ M \to X$$

Four customers of the company located at A, B, C and D want to transport nearly the same size of load to their final destinations. The final destinations are at a distance of 600, 300, 1000 and 500 kms from the loading points A, B, C and D, respectively.

Distances in kms. between location of the aircraft and the loading points are as follows .

Loading points

		A	B	C	D
	J	300	200	400	100
Present location	K	300	100	300	300
of aircraft	L	400	100	100	500
	M	200	200	400	200

Determine the allocations which minimize the total cost of transportation.

[**Ans.** $J \to D$; $K \to B$, $L \to C$, $M \to A$].

6. The jobs A, B, C are to be assigned three machines X, Y, Z. The processing costs (Rs.) are as given in the matrix shown below. Find the allocation which will minimize the overall processing cost.

Machine

		X	Y	Z
	A	19	28	31
Job	B	11	17	16
	C	12	15	13

[**Ans.** $A-X$; $B-Y$, $C-Z$].

7. A project work consists of four major jobs for which four contractors have submitted tenders. The tender amounts quoted in lakhs of rupees are given in the matrix below :

Job

		a	b	c	d
	1	10	24	30	15
Contractor	2	16	22	28	12
	3	12	20	32	10
	4	9	26	34	16

Find the assignment which minimizes the total cost of project [each contract has to be assigned at least one job].

[**Ans.** Three alternative assignments are : (i) $1 \to b$, $2 \to c$, $3 \to d$, $4 \to a$ (ii) $1 \to c$, $2 \to b$, $3 \to d$, $4 \to a$; (iii) $1 \to c$, $2 \to d$, $3 \to b$, $4 \to a$; min. cost = Rs. 71,00,000].

8. A company is producing a single product and selling it through five agencies situated in different cities. All of a sudden, there is a demand for the product in another five cities not having any agency of the comapny. The company is faced with the problem of deciding on how to assign the existing agencies to despatch the product to needy cities in such a way that the total travelling distance is minimized. The distance between the surplus and deficit cities (in kilometers) is given by

Deficit city

Surplus city	A'	B'	C'	D'	E'
A	10	5	9	18	11
B	13	19	6	12	14
C	3	2	4	4	5
D	18	9	12	17	15
E	11	6	14	19	10

Determine the optimum assignment schedule.

[Ans. $A \to A'$, $B \to C'$, $C \to D'$, $D \to B'$, $E \to E'$, $z^* = 39$ km.]

9. Find the minimum cost solution for the 5 × 5 assignment problem whose cost coefficients are as given below:

	1	2	3	4	5
1	−2	−4	−8	−6	−1
2	0	−9	−5	−5	−4
3	−3	−8	0	−2	−6
4	−4	−3	−1	0	−3
5	−9	−5	−9	−9	−5

[Ans. $1 \to 3, 2 \to 2, 3 \to 5, 4 \to 4, 5 \to 1$ or $1 \to 4, 2 \to 2, 3 \to 3, 4 \to 5, 5 \to 1$, $z^* = 36$]

10. A company has 4 machines of which to do 3 jobs. Each job can be assigned to one and only one machine. The cost of each job on each machine is given in the following table:

Machine

Job	W	X	Y	Z
A	18	24	28	32
B	8	13	17	18
C	10	15	19	22

What are the job-assignments which will minimize the cost?

[Ans. $A \to W$, $B \to X$, $C \to Y$, or $A \to W$, $B \to Y$, $C \to X$, $z^* = 50$].

11. Six wagons are available at six stations *A, B, C, D, E* and *F*. These are required at stations *I, II, III, IV, V* and *VI*. The mileage between various stations is given by the following table:

	I	II	III	IV	V	VI
A	20	23	18	10	16	20
B	50	20	17	16	15	11
C	60	30	40	55	8	7
D	6	7	10	20	100	9
E	18	19	28	17	60	70
F	9	10	20	30	40	55

How should the wagons be transported in order to minimize the total mileage covered.
[Ans. $A \to IV$, $B \to VI$, $C \to V$, $D \to III$, $E \to I$, $F \to II$, total mileage = 66].

12. The owner of a small machine shop has four machinists avilable to assign to jobs for the day. five jobs offered with the expected profit (in Rs.) for each machinist on each job being as follows.

Job

Machinist	A	B	C	D	E
1	6.20	7.80	5.00	10.10	8.20
2	7.10	8.40	6.10	7.30	5.90
3	8.70	9.20	11.10	7.10	8.10
4	4.80	6.40	8.70	7.70	8.00

Find the assignment of machinists to jobs that will result in a maximum profit. Which job should be declined?
[Ans. $1 \to D, 2 \to B, 3 \to C, 4 \to E, 5 \to A$, min. cost = Rs. 37·60. Job A should be declined].

13. A computer centre has got three programmers. The centre needs three application programmes to be developed. The Head of the computer centre, after studying carefully the programmes to be developed, estimates the computer time in minutes required by the experts to the application programmes as follows :

		Programmers		
		A	B	C
Programmes	1	120	100	80
	2	70	90	110
	3	110	140	120

Assign the programmers to the programmes in such a way that the total computer time is least.
[Ans. $1 \to C, 2 \to B, 3 \to A$]

14. The Research & Development department of an organization is having four major jobs to be completed in ensuing financial period. There are four subgroups who can work on these jobs. Because of the technical nature of problems and heterogeneous combination of groups the cost of completing the work is different for different groups as shown in the following table :

		Job			
		1	2	3	4
Group	I	20	22	28	15
	II	16	20	12	13
	III	19	23	14	25
	IV	10	16	12	10

Allocate the jobs to the groups in such a way that the R & D budget is minimum.
[Ans. $I \to 2, II \to 4, III \to 3, IV \to 1$; $I \to 4, II \to 2, III \to 3, IV \to 1$; *min* cost = Rs. 59].

15. Four engineers are available to design four projects. Engineer 2 is not competent to design the project *B*. Given the following time estimates needed by each engineer to design a given project, find how should the engineers be assigned to projects so as to minimize the total design time of four projects.

		Projects.			
		A	B	C	D
Engineer	1	12	10	10	8
	2	14	Not Suitable	15	11
	3	6	10	16	4
	4	8	10	9	7

[Ans. $1 \to B, 2 \to D, 3 \to A, 4 \to C$; total time = 36 hours.]

16. The owner of a small machine shop has four machinists available to assign to jobs for the day. Five jobs are offered with expected profit for each machinist on each job as follows :

	A	B	C	D	E
1	62	78	50	101	82
2	71	84	61	73	59
3	87	92	111	71	81
4	48	64	87	77	80

Find by using the assignment method, the assignment of machinists to jobs that will result in a maximum profit. Which job should be declined ?
[Ans. $1 \to D, 2 \to B, 3 \to C, 4 \to E$; max. profit = Rs . 376 .]

17. A company is faced with the problem of assigning 4 machines to 6 different jobs (one machine to one job only). The profits are estimated as follows :

		Machine			
		A	B	C	D
Job	1	3	6	2	6
	2	7	1	4	4
	3	3	8	5	8
	4	6	4	3	7
	5	5	2	4	3
	6	5	7	6	4

Solve the problem to maximize the total profit.
[Ans. $2 \to A, 3 \to B, 4 \to D, 6 \to C$, max. profit = 28 .]

18. Find the optimal assingment for the given assignment.

		Machine 1	2	3
	1	5	7	9
Job	2	14	10	12
	3	15	13	16

[I.A.S (Maths.) 99]

19. (a) The solution to assignment problem is inherently degenerate. Explain. **[VTU (BE Mech.) 2003]**

(b) An Air Transport Co. picks up and delivers freight where customers require. The Co. has two types of air craft *X* and *Y* with equal loading capacities but different operating costs as shown :

Type of Air craft	Cost per KM in Rs.	
	Empty	loaded
X	1.00	2.00
Y	2.00	3.00

The present four locations of the air craft which the Co. is having are :

$$J \to X, K \to Y, L \to Y \text{ and } M \to X$$

Four customers of the Co. located at *A, B, C* and *D* want to transport nearly the same load to their final destinations. The final destinations are at a distance of 600, 300, 100 and 500 KMs from loading points. *A, B, C* and *D* respectively. Distance between locations of air craft and loading points are as follows :

	A	B	C	D
J	300	200	400	100
K	300	100	300	300
L	400	100	100	400
M	200	200	300	200

Determine the optimum allocation and total cost. **[VTU (BE Mech.) 2003]**

MODEL OBJECTIVE QUESTIONS

1. An assignment problem is considered as a particular case of a transportation problem, because
(a) the number of rows equals the number of columns. (b) all $x_{ij} = 0$ or 1.
(c) all rim conditions are 1. (d) all of the above.
2. An optimal assignment requires that the maximum number of lines which can be drawn through squares with zero opportunity cost be equal to the number of
(a) rows or columns. (b) rows and columns.
(c) rows + column–1. (d) none of the above.
3. While solving assignment problem. an activity is assigned to a resource through a square with zero opportunity cost because the objective is to
(a) minimize total cost of assignment. (b) reduce the cost of assignment to zero.
(c) reduce the cost of that particular assignment to zero. (d) all of the above.
4. The method used for solving an assignment problem is called
(a) reduced matrix method. (b) MODI method. (c) Hungarian method. (d) none of the above.
5. The purpose of a dummy row or column in an assignment problem is to
(a) obtain balance between total activities and total resources.
(b) prevent a solution from becoming degenerate.
(c) provide the means of representing a dummy problem.
(d) none of the above.
6. Maximization assignment problem is tansformed into a minimization problem by
(a) adding each entry in a column from the maximum value in that column.
(b) subtracting each entry in a column from the maximum value in that column.
(c) subtracting each entry in the table from the maximum value in that table.
(d) any one of the above.
7. If there were *n* workers and *n* jobs, there would be
(a) n ! solutions. (b) $(n-1)$! solutions. (c) $(n\,!)^n$ solutions. (d) n solutions.
8. An assignment problem can be solved by
(a) simplex method. (b) transportation method. (c) both (a) and (b). (d) none of the above.

9. For a salesman who has to visit n cities. following are the ways of his tour plan
   (a) $n!$. (b) $(n+1)!$. (c) $(n-1)!$. (d) n.
10. The assignment problem
   (a) requires that only one activity be assigned to each resource.
   (b) is a special case of transportation problem.
   (c) can be used to maximize resources.
   (d) all of the above.
11. The assignment algorithm is applicable to which of the following combined situations for the purpose of improving productivity ?
   1. Identification of sales force-market, 2. Scheduling of operator machine, 3. Fixing machine location.
   Select the correct answer using the codes given below.
   Codes : (a) 1, 2 and 3 (b) 1 and 3 (c) 2 and 3 (d) 1 and 2.
   [IES (Mech.) 1998]

Answers

1. (d)	2. (a)	3. (a)	4. (c)	5. (a)	6. (c)	7. (a)	8. (c)	9. (c)
10. (d)	11. (c)							

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8

NETWORK MODELS : PERT/CPM

8.1. INTRODUCTION

A project defines a combination of interrelated activities which must be executed in a certain order before the entire task can be completed. The activities are interrelated in a logical sequence in such a way that some activities cannot start until some others are completed. An activity in a project is usually viewed as job requiring time and resources for its completion. Until recently, planning was seldom used in the design phase. As the technological development took place at a very rapid speed and the designs become more complex with more inter-departmental dependence and interaction, the need for planning in the development phase become inevitable.

Until five decades ago, the best known 'planning tool' was the so called *Gantt bar chart* which specifies the start and finish times for each activity on a horizontal time-scale, but the disadvantage is the interdependency between different activities (that mainly controls the progress of the project) which cannot be determined from the bar chart. Growing complexities of modern projects have demanded more systematic and effective planning techniques with the objective of optimizing the efficiency of executing the project. Efficiency implies effecting the utmost reduction in the time required to complete the project while accounting for economic feasibility of using available resources. Project management has evolved as a new field with the development of *two* 'analytic' techniques for planning, scheduling and controlling of projects. These are the *Critical Path Method (CPM)* and the *Project Evaluation and Review Technique (PERT).*

8.2. HISTORICAL DEVELOPMENT OF PERT/CPM TECHNIQUES

In 1956-58, above two techniques were developed by two different groups almost simultaneously. CPM was developed by Walker from E.L. du pont de Nemours Company to solve project scheduling problems and was later extended to a more advanced status by Mauchly Associates. During the same time, PERT was developed by the team of engineers working on the polar's Missile programme of US Navy. This was a large project involving many departments and there were many activities about which they had a very little information about the duration of the project. Under such conditions, the project was to be completed within a specified time. To coordinate activities of various departments, this group used PERT and devised the technique independent of CPM.

The methods are essentially network-oriented techniques using the same principle. PERT and CPM are basically time-oriented methods in the sense that they both lead to the determination of a time schedule for the project. The significant difference between two approaches is that the time estimates for the different activities in CPM were assumed to be deterministic while in PERT these were described probabilistically. Now a days, PERT and CPM actually comprise one technique and the differences, if any are only historical. Therefore, these techniques are referred to as '*project scheduling*' techniques.

Q. Distinguish between PERT and CPM techniques. [VTU (BE Mech.) 2002]

8.3. APPLICATIONS OF PERT / CPM TECHNIQUES

These methods have been applied to a wide variety of problems in industries and have found acceptance even in government organizations.